

CONSTRUCTION OF TWO-BUBBLE SOLUTIONS FOR ENERGY-CRITICAL WAVE EQUATIONS

JACEK JENDREJ

ABSTRACT. We construct pure two-bubbles for some energy-critical wave equations, that is solutions which in one time direction approach a superposition of two stationary states both centered at the origin, but asymptotically decoupled in scale. Our solution exists globally, with one bubble at a fixed scale and the other concentrating in infinite time, with an error tending to 0 in the energy space. We treat the cases of the power nonlinearity in space dimension 6, the radial Yang-Mills equation and the equivariant wave map equation with equivariance class $k \geq 3$. The concentration speed of the second bubble is exponential for the first two models and a power function in the last case.

1. INTRODUCTION

1.1. Energy critical NLW. We consider the energy critical wave equation in space dimension $N = 6$:

$$(1.1) \quad \begin{cases} (\partial_t^2 - \Delta)u(t, x) = |u(t, x)| \cdot u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^6, \\ (u(t_0, x), \partial_t u(t_0, x)) = (u_0(x), \dot{u}_0(x)). \end{cases}$$

The *energy functional* associated with this equation is defined for $\mathbf{u}_0 = (u_0, \dot{u}_0) \in \mathcal{E} := \dot{H}^1(\mathbb{R}^6) \times L^2(\mathbb{R}^6)$ by the formula

$$E(\mathbf{u}_0) := \int \frac{1}{2}|\dot{u}_0|^2 + \frac{1}{2}|\nabla u_0|^2 - F(u_0) \, dx,$$

where $F(u_0) := \frac{1}{3}|u_0|^3$. Note that $E(\mathbf{u}_0)$ is well-defined due to the Sobolev Embedding Theorem. The differential of E is $DE(\mathbf{u}_0) = (-\Delta u_0 - f(u_0), \dot{u}_0)$, where $f(u_0) = |u_0| \cdot u_0$, hence we can rewrite equation (1.1) as

$$(1.2) \quad \begin{cases} \partial_t \mathbf{u}(t) = J \circ DE(\mathbf{u}(t)), \\ \mathbf{u}(t_0) = \mathbf{u}_0 \in \mathcal{E}. \end{cases}$$

Here, $J := \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$ is the natural symplectic structure.

Equation (1.1) is locally well-posed in the space \mathcal{E} , see for example Ginibre, Soffer and Velo [17], Shatah and Struwe [36] (the defocusing case), as well as a complete review of the Cauchy theory in Kenig and Merle [24] (for $N \in \{3, 4, 5\}$) and Bulut, Czubak, Li, Pavlović and Zhang [4] (for $N \geq 6$). By “well-posed” we mean that for any initial data $\mathbf{u}_0 \in \mathcal{E}$ there exists $\tau > 0$ and a unique solution in some subspace of $C([t_0 - \tau, t_0 + \tau]; \mathcal{E})$, and that this solution is continuous with respect to the initial data. By standard arguments, there exists a maximal time of existence (T_-, T_+) , $-\infty \leq T_- < t_0 < T_+ \leq +\infty$, and a unique solution $\mathbf{u} \in C((T_-, T_+); \mathcal{E})$. If $T_+ < +\infty$, then $\mathbf{u}(t)$ leaves every compact subset of \mathcal{E} as t approaches T_+ . A crucial property of the solutions of (1.1) is that the energy E is a conservation law.

In this paper we always assume that the initial data are radially symmetric. This symmetry is preserved by the flow.

For functions $v \in \dot{H}^1$, $\dot{v} \in L^2$, $\mathbf{v} = (v, \dot{v}) \in \mathcal{E}$ and $\lambda > 0$, we denote

$$v_\lambda(x) := \frac{1}{\lambda^2} v\left(\frac{x}{\lambda}\right), \quad \dot{v}_\lambda(x) := \frac{1}{\lambda^3} \dot{v}\left(\frac{x}{\lambda}\right), \quad \mathbf{v}_\lambda(x) := (v_\lambda, \dot{v}_\lambda).$$

A change of variables shows that

$$E((\mathbf{u}_0)_\lambda) = E(\mathbf{u}_0).$$

Equation (1.1) is invariant under the same scaling: if $\mathbf{u}(t) = (u(t), \dot{u}(t))$ is a solution of (1.1) and $\lambda > 0$, then $t \mapsto \mathbf{u}(t_0 + t/\lambda)_\lambda$ is also a solution with initial data $(\mathbf{u}_0)_\lambda$ at time $t = 0$. This is why equation (1.1) is called *energy-critical*.

A fundamental object in the study of (1.1) is the family of stationary solutions $\mathbf{u}(t) \equiv \pm \mathbf{W}_\lambda = (\pm W_\lambda, 0)$, where

$$W(x) = \left(1 + \frac{|x|^2}{24}\right)^{-2}.$$

The functions $\pm W_\lambda$ are called *ground states* or *bubbles* (of energy). They are the only radially symmetric solutions of the critical elliptic problem

$$(1.3) \quad -\Delta u - f(u) = 0.$$

The functions W_λ are, up to translation, the only positive solutions of (1.3). The ground states achieve the optimal constant in the critical Sobolev inequality, which was proved by Aubin [1] and Talenti [40]. They are the “mountain passes” for the potential energy.

Kenig and Merle [24] exhibited the special role of the ground states \mathbf{W}_λ as the *threshold elements* for nonlinear dynamics of the solutions of (1.1) in space dimensions $N = 3, 4, 5$, which is believed to be a general feature of dispersive equations (the so-called *Threshold Conjecture*). Another major problem in the field is the *Soliton Resolution Conjecture*, which predicts that a bounded (in an appropriate sense) solution decomposes asymptotically into a sum of energy bubbles at different scales and a radiation term (a solution of the linear wave equation). This was proved for the radial energy-critical wave equation in dimension $N = 3$ by Duyckaerts, Kenig and Merle [15], following the earlier work of the same authors [14], where such a decomposition was proved only for a sequence of times (this last result was generalized to any odd dimension by Rodriguez [35]).

It is natural to examine the dynamics of the solutions of (1.1) in a neighborhood (in the energy space) of the family of the ground states. In dimension $N = 3$ this was done by Krieger, Schlag and Tataru [27], who showed that such solutions can blow up in finite time (by concentration of the bubble), see also [12], [25], [11], [19], [21] for related results.

In view of the rich dynamics in a neighborhood of one bubble, it was expected that solutions behaving asymptotically as a superposition of many (at least two) bubbles exist, in other words that the result of [15] is essentially optimal. We prove that it is the case when $N = 6$:

Theorem 1. *There exists a solution $\mathbf{u} : (-\infty, T_0] \rightarrow \mathcal{E}$ of (1.1) such that*

$$\lim_{t \rightarrow -\infty} \|\mathbf{u}(t) - (\mathbf{W} + \mathbf{W}_{\frac{1}{\kappa}e^{-\kappa|t|}})\|_{\mathcal{E}} = 0, \quad \text{with } \kappa := \sqrt{\frac{5}{4}}.$$

Remark 1.1. More precisely, we will prove that

$$\|\mathbf{u}(t) - (\mathbf{W} + \mathbf{W}_{\frac{1}{\kappa}e^{-\kappa|t|}})\|_{\mathcal{E}} \leq C_1 \cdot e^{-\frac{1}{2}\kappa|t|}$$

for some constant $C_1 > 0$.

Remark 1.2. We construct here *pure* two-bubbles, that is the solution approaches a superposition of two stationary states, with no energy transformed into radiation. By the conservation of energy and the decoupling of the two bubbles, we necessarily have $E(\mathbf{u}(t)) = 2E(\mathbf{W})$. Pure one-bubbles cannot concentrate and are completely classified, see [16].

Remark 1.3. It was proved in [22], in any dimension $N \geq 3$, that there exist no solutions $\mathbf{u}(t) : [t_0, T_+) \rightarrow \mathcal{E}$ of (1.1) such that $\|\mathbf{u}(t) - (\mathbf{W}_{\mu(t)} - \mathbf{W}_{\lambda(t)})\|_{\mathcal{E}} \rightarrow 0$ with $\lambda(t) \ll \mu(t)$ as $t \rightarrow T_+ \leq +\infty$.

Remark 1.4. In any dimension $N > 6$ one can expect an analogous result with concentration rate $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$.

Remark 1.5. In the context of the harmonic map heat flow, Topping [41] proved the existence of towers of bubbles for a well chosen target manifold, see also a non-existence result of van der Hout [20].

Let us resume the overall strategy of the proof, which is based on the previous paper of the author [21].

In Section 2 we construct an appropriate approximate solution $\varphi(t)$. We present first a formal computation which allows to predict the concentration rate and explains why the proof fails in dimension $N \in \{3, 4, 5\}$. It highlights also the role of the *strong interaction* between the two bubbles (by “strong” we mean “significantly altering the dynamics”; [31] provides an example of this phenomenon in a different context). Then we give a precise definition of the approximate solution and prove bounds on its error.

In Section 3 we build a sequence $\mathbf{u}_n : [T_n, T_0] \rightarrow \mathcal{E}$ of solutions of (1.1) with $T_n \rightarrow -\infty$ and $\mathbf{u}_n(t)$ close to a two-bubble solution for $t \in [T_n, T_0]$. Taking a weak limit finishes the proof. This type of argument goes back to the works of Merle [32] and Martel [29]. The heart of the analysis is to obtain uniform energy bounds for the sequence \mathbf{u}_n . To this end we follow the approach of Raphaël and Szeftel [33], that is we prove bootstrap estimates involving an energy functional with a virial-type correction term. This correction is designed to cancel some terms related to the concentration of the bubble $\mathbf{W}_{\lambda(t)}$. It has to be localized in an appropriate way, so that it does not “see” the other bubble. Finally, in order to deal with the linear instabilities of the flow, we use a classical topological (“shooting”) argument.

1.2. Critical wave maps. We consider the wave map equation from the 2+1-dimensional Minkowski space (the energy-critical case) to \mathbb{S}^2 . We will consider solutions with k -equivariant symmetry, in which case the problem is reduced to the following scalar equation:

$$(1.4) \quad \begin{cases} \partial_t^2 u(t, r) = \partial_r^2 u(t, r) + \frac{1}{r} \partial_r u(t, r) - \frac{k^2}{2r^2} \sin(2u(t, r)), \\ (u(t_0, r), \partial_t u(t_0, r)) = (u_0(r), \dot{u}_0(r)), \quad t, t_0 \in \mathbb{R}, r \in (0, +\infty). \end{cases}$$

For a presentation of the geometric content of this equation, one can consult [37]. Here we will regard (1.4) as a scalar semilinear problem.

We define the space \mathcal{H} as the completion of $C_0^\infty((0, +\infty))$ for the norm

$$\|v\|_{\mathcal{H}}^2 := 2\pi \int_0^{+\infty} (|\partial_r v(r)|^2 + |\frac{k}{r} v(r)|^2) r dr.$$

We will work in the energy space $\mathcal{E} := \mathcal{H} \times L^2$. Equation (1.4) can be written in the form (1.2) with the energy functional E defined for $\mathbf{u}_0 = (u_0, \dot{u}_0) \in \mathcal{E}$ by the formula

$$E(\mathbf{u}_0) := \pi \int_0^{+\infty} ((\dot{u}_0)^2 + (\partial_r u_0)^2 + \frac{k^2}{r^2} (\sin(u_0))^2) r dr.$$

The Cauchy theory in the energy space has been established by Shatah and Tahvildar-Zadeh [38]. Note that $u_0 \in \mathcal{H}$ forces $\lim_{r \rightarrow +\infty} u_0(r) = 0$, but we could just as well consider states of finite energy such that $\lim_{r \rightarrow +\infty} u_0(r) \in \pi\mathbb{Z}$, see [9, 8] for details.

The stationary solutions $W_\lambda(r) := 2 \arctan\left(\left(\frac{r}{\lambda}\right)^k\right)$ play a fundamental role in the study of (1.4). They are the harmonic maps of topological degree k . We will write $W(r) := W_1(r) = 2 \arctan(r^k)$

and $\Lambda W(r) := -\frac{\partial}{\partial \lambda} W_\lambda|_{\lambda=1} = \frac{2k}{r^k + r^{-k}}$. Note that $W \notin \mathcal{H}$ precisely because of the fact that $W(r) \rightarrow \pi$ as $r \rightarrow +\infty$.

The possibility of concentration of a harmonic map at the origin was first observed numerically by Bizoń, Chmaj and Tabor [3]. Struwe [39] proved that if the blow-up occurs, then \mathbf{W} is the blow-up profile (for a sequence of times). The dynamics in a neighborhood of a harmonic map was studied by Krieger, Schlag and Tataru [26], who constructed blow-up solutions in the energy space with the concentration rate $\lambda(t) \sim (T_+ - t)^{1+\nu}$ for all $\nu > \frac{1}{2}$. This behavior is expected to be highly unstable. Rodnianski and Sterbenz [34] constructed stable blow-up solutions, giving the first (partial) rigorous explanation of the surprising numerical results mentioned above. In the case $k = 1$, Côte [7] proved that any solution decomposes, for a sequence of times tending to the final (finite or infinite) time of existence, as a sum of a finite number of harmonic maps at different scales and a radiation term. A generalization of this result, including all the cases considered in this paper, was recently obtained by Jia and Kenig [23]. Motivated by these works, we prove the following result.

Theorem 2. *Fix $k > 2$. There exists a solution $\mathbf{u} : (-\infty, T_0] \rightarrow \mathcal{E}$ of (1.4) such that*

$$\lim_{t \rightarrow -\infty} \|\mathbf{u}(t) - (-\mathbf{W} + \mathbf{W}_{\frac{k-2}{2\kappa}(\kappa|t|)^{-\frac{2}{k-2}}})\|_{\mathcal{E}} = 0, \quad \text{with } \kappa := \frac{k-2}{2} \left(\frac{8k}{\pi} \sin\left(\frac{\pi}{k}\right) \right)^{\frac{1}{k}}.$$

Remark 1.6. More precisely, we will prove that

$$\|\mathbf{u}(t) - (-\mathbf{W} + \mathbf{W}_{\frac{k-2}{2\kappa}(\kappa|t|)^{-\frac{2}{k-2}}})\|_{\mathcal{E}} \leq C_1 \cdot |t|^{-\frac{1}{2(k-2)}}.$$

Remark 1.7. The constructed solution is a *pure* two-bubble, hence by the conservation of energy $E(\mathbf{u}(t)) = 2E(\mathbf{W})$, and it is clear that it has the homotopy degree 0. In the case of equivariant class $k = 1$, Côte, Kenig, Lawrie and Schlag [8] showed that any degree 0 initial data of energy $< 2E(\mathbf{W})$ leads to dispersion (the proof is expected to generalize to all equivariance classes). Theorem 3 gives the first example of a non-dispersive solution at the threshold energy.

Note that pure two-bubbles of homotopy degree $2k$ (hence of type bubble-bubble and not bubble-antibubble) do not exist because the energy of such a map has to be $> 2E(\mathbf{W})$. This is similar to the case of opposite signs for (1.1), see Remark 1.3.

Remark 1.8. I believe that the proof can be adapted to deal with a more general equation $\partial_t^2 u = \partial_r^2 u + \frac{1}{r} \partial_r u - \frac{1}{r^2} (gg')(u)$ with g satisfying the assumptions of [9] and $g'(0) \in \{3, 4, 5, \dots\}$.

1.3. Critical Yang-Mills. Finally, we consider the radial Yang-Mills equation in dimension 4 (which is the energy-critical case):

$$(1.5) \quad \begin{cases} \partial_t^2 u(t, r) = \partial_r^2 u(t, r) + \frac{1}{r} \partial_r u(t, r) - \frac{4}{r^2} u(t, r)(1 - u(t, r))(1 - \frac{1}{2}u(t, r)), \\ (u(t_0, r), \partial_t u(t_0, r)) = (u_0(r), \dot{u}_0(r)), \quad t, t_0 \in \mathbb{R}, \quad r \in (0, +\infty). \end{cases}$$

For a derivation of this equation and further comments, see for instance [6]. Equation (1.5) can be written in the form (1.2) with the energy functional E defined for $\mathbf{u}_0 = (u_0, \dot{u}_0) \in \mathcal{E}$ by the formula

$$E(\mathbf{u}_0) := \pi \int_0^{+\infty} ((\dot{u}_0)^2 + (\partial_r u_0)^2 + \frac{1}{r^2} (u_0(2 - u_0))^2) r dr.$$

The stationary solutions of (1.5) are $W_\lambda(r) := \frac{2r^2}{\lambda^2 + r^2}$. We denote $W(r) := W_1(r) = \frac{2r^2}{1+r^2}$ and $\Lambda W(r) := -\frac{\partial}{\partial \lambda} W_\lambda|_{\lambda=1} = \frac{4}{(r+r^{-1})^2}$.

Theorem 3. *There exists a solution $\mathbf{u} : (-\infty, T_0] \rightarrow \mathcal{E}$ of (1.5) such that*

$$\lim_{t \rightarrow -\infty} \|\mathbf{u}(t) - (-\mathbf{W} + \mathbf{W}_{\frac{1}{\kappa}e^{-\kappa|t|}})\|_{\mathcal{E}} = 0, \quad \text{with } \kappa := 2\sqrt{3}.$$

Remark 1.9. More precisely, we will prove that

$$\|\mathbf{u}(t) - (-\mathbf{W} + \mathbf{W}_{\frac{1}{\kappa}e^{-\kappa|t|}})\|_{\mathcal{E}} \leq C_1 \cdot e^{-\frac{1}{2}\kappa|t|},$$

where $\Lambda W := -\frac{\partial}{\partial \lambda} W_\lambda|_{\lambda=1}$ and $C_1 > 0$ is a constant.

Remark 1.10. The case of wave maps in the equivariance class $k = 2$ should be very similar.

Remark 1.11. The energy $2E(\mathbf{W})$ is the threshold energy for a non-dispersive behavior for solutions with topological degree 0, see [28].

1.4. Structure of the paper. In Sections 2 and 3 we give a detailed proof of Theorem 1. In Section 4 we treat the case of the Yang-Mills equation. We skip these parts of the proof where the arguments of Sections 2 and 3 are directly applicable. Section 5 is devoted to the wave maps equation. The main difference with respect to Section 4 is that the characteristic length of the concentrating bubble is now a power of $|t|$ and not an exponential. Nevertheless, large parts of the previous proofs extend to this case and are skipped. It is conceivable that one could propose a unified, more general framework of the proof, encompassing all the cases under consideration. Appendix A is devoted to some elements of the local Cauchy theory needed in the proofs.

1.5. Notation. The bracket $\langle \cdot, \cdot \rangle$ denotes the distributional pairing and the scalar product in the spaces L^2 and $L^2 \times L^2$.

For positive quantities m_1 and m_2 we write $m_1 \lesssim m_2$ if $m_1 \leq C m_2$ for some constant $C > 0$ and $m_1 \sim m_2$ if $m_1 \lesssim m_2 \lesssim m_1$.

We denote χ a standard C^∞ cut-off function, that is $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$ and $0 \leq \chi(x) \leq 1$ for $1 \leq |x| \leq 2$.

2. CONSTRUCTION OF AN APPROXIMATE SOLUTION – THE NLW CASE

2.1. Inverting the linearized operator. Linearizing (1.1) around \mathbf{W} , $\mathbf{u} = \mathbf{W} + \mathbf{h}$, one obtains

$$\partial_t \mathbf{h} = J \circ D^2 E(\mathbf{W}) \mathbf{h} = \begin{pmatrix} 0 & \text{Id} \\ -L & 0 \end{pmatrix} \mathbf{h},$$

where L is the Schrödinger operator

$$Lh := (-\Delta - f'(W))h = (-\Delta - 2W)h.$$

We introduce the following notation for the generators of the \dot{H}^1 -critical and the L^2 -critical scale change:

$$\Lambda := 2 + x \cdot \nabla, \quad \Lambda_0 := 3 + x \cdot \nabla.$$

This is coherent with the definition of ΛW . Notice that $L(\Lambda W) = \frac{\partial}{\partial \lambda} W_\lambda|_{\lambda=1} (-\Delta W_\lambda - f(W_\lambda)) = 0$.

We fix $\mathcal{Z} \in C_0^\infty$ such that

$$\langle \mathcal{Z}, \Lambda W \rangle > 0, \quad \langle \mathcal{Z}, \mathcal{Y} \rangle = 0.$$

We will use this function to define appropriate orthogonality conditions.

We denote also

$$(2.1) \quad \kappa := \left(-\frac{\langle \Lambda W, f'(W) \rangle}{\langle \Lambda W, \Lambda W \rangle} \right)^{\frac{1}{2}} = \sqrt{\frac{5}{4}}.$$

Lemma 2.1. *There exist radial rational functions $P(x), Q(x) \in C^\infty(\mathbb{R}^6)$ such that*

$$(2.2) \quad LP = \kappa^2 \Lambda W + f'(W), \quad LQ = -\Lambda_0 \Lambda W,$$

$$(2.3) \quad \langle \mathcal{Z}, P \rangle = \langle \mathcal{Z}, Q \rangle = 0,$$

$$(2.4) \quad P(x) \sim |x|^{-2}, \quad Q(x) \sim |x|^{-2} \quad \text{as } |x| \rightarrow +\infty.$$

Proof. By a direct computation one checks that the functions

$$\begin{aligned} \tilde{P}(x) &:= \left(1 + \frac{|x|^2}{24}\right)^{-3} \cdot \left(1 - 10 \cdot \frac{|x|^2}{24} - 3 \cdot \left(\frac{|x|^2}{24}\right)^2\right), \\ \tilde{Q}(x) &:= \left(1 + \frac{|x|^2}{24}\right)^{-3} \cdot \left(1 + 11 \cdot \frac{|x|^2}{24} - 12 \cdot \left(\frac{|x|^2}{24}\right)^2\right) \end{aligned}$$

satisfy (2.2). Adding suitable multiples of ΛW to both functions we obtain P and Q satisfying (2.3). The formulas defining \tilde{P} and \tilde{Q} directly imply (2.4). \square

Remark 2.2. Note that (2.4) is closely related to the Fredholm conditions $\langle \Lambda W, \kappa^2 \Lambda W + f'(W) \rangle = 0$ and $\langle \Lambda W, -\Lambda_0 \Lambda W \rangle = 0$, see Lemma 5.1 or [21, Proposition 2.1] for a more systematic presentation.

2.2. Formal computation. The usual method of performing a formal analysis of blow-up solutions is to search a series expansion with respect to a small scalar parameter depending on time and converging to 0 at blow-up. In our case the blow-up time is $-\infty$. If $u(t) \simeq W + W_{\lambda(t)}$, then $\partial_t u(t) \simeq -\lambda'(t) \Lambda W_{\lambda(t)}$, hence

$$\mathbf{u}(t) \simeq (W + W_{\lambda(t)}, 0) - \lambda'(t) \cdot (0, \Lambda W_{\lambda(t)}) = \mathbf{W} + \mathbf{U}_{\lambda(t)}^{(0)} + b(t) \cdot \mathbf{U}_{\lambda(t)}^{(1)},$$

with $b(t) := \lambda'(t)$, $\mathbf{U}^{(0)} := (W, 0)$ and $\mathbf{U}^{(1)} := (0, -\Lambda W)$. This suggests considering $b(t) = \lambda'(t)$ as the small parameter with respect to which the formal expansion should be sought. Hence, we make the ansatz

$$\mathbf{u}(t) = \mathbf{W} + \mathbf{U}_{\lambda(t)}^{(0)} + b(t) \cdot \mathbf{U}_{\lambda(t)}^{(1)} + b(t)^2 \cdot \mathbf{U}_{\lambda(t)}^{(2)},$$

and try to find the conditions under which a satisfactory candidate for $\mathbf{U}^{(2)} = (U^{(2)}, \dot{U}^{(2)})$ can be proposed. Neglecting irrelevant terms and replacing $\lambda'(t)$ by $b(t)$, we compute

$$\partial_t^2 u(t) = -b'(t)(\Lambda W)_{\lambda(t)} + \frac{b(t)^2}{\lambda(t)}(\Lambda_0 \Lambda W)_{\lambda(t)} + \text{lot}.$$

On the other hand, using the fact that $f(W + W_\lambda) = f(W) + f(W_\lambda) + f'(W_\lambda)W \simeq f(W) + f(W_\lambda) + f'(W_\lambda)$ for $\lambda \ll 1$ and $f'(W_\lambda) = \lambda f'(W)_{\lambda}$, we get

$$\Delta u(t) + f(u(t)) = -\frac{b(t)^2}{\lambda(t)}(LU^{(2)})_{\lambda(t)} + \lambda(t)f'(W)_{\lambda(t)} + \text{lot}.$$

We discover that, formally at least, we should have

$$(2.5) \quad LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}(b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)).$$

Lemma 2.1 shows that if $b'(t) = \kappa^2 \lambda(t)$, then equation (2.5) has a decaying regular solution $U^{(2)} = Q + \frac{\lambda(t)^2}{b(t)^2}P$. The *formal parameter equations*

$$\lambda'(t) = b(t), \quad b'(t) = \kappa^2 \lambda(t)$$

have a solution

$$(\lambda_{\text{app}}(t), b_{\text{app}}(t)) = \left(\frac{1}{\kappa} e^{-\kappa|t|}, e^{-\kappa|t|}\right), \quad t \leq T_0 < 0.$$

In any space dimension N , ignoring the problems related to slow decay of W , a similar analysis would yield $b'(t) = \kappa^2 \lambda(t)^{\frac{N-4}{2}}$. For $N < 6$ this leads to a finite time blow-up, which was studied in [21] for $N = 5$. For $N > 6$, we obtain a global solution $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$, see Remark 1.4.

2.3. Bounds on the error of the ansatz. Let $I = [T, T_0]$ be a time interval, with $T \leq T_0 < 0$ and $|T_0|$ large. Let $\lambda(t)$ and $\mu(t)$ be C^1 functions on $[T, T_0]$ such that

$$(2.6) \quad \lambda(T) = \frac{1}{\kappa} e^{-\kappa|T|}, \quad \mu(T) = 1,$$

$$(2.7) \quad \frac{8}{9\kappa} e^{-\kappa|t|} \leq \lambda(t) \leq \frac{9}{8\kappa} e^{-\kappa|t|}, \quad \frac{8}{9} \leq \mu(t) \leq \frac{9}{8}.$$

We define the *approximate solution* $\varphi(t) = (\varphi(t), \dot{\varphi}(t)) : [T, T_0] \rightarrow \mathcal{E}$ by the formula

$$\begin{aligned} \varphi(t) &:= W_{\mu(t)} + W_{\lambda(t)} + S(t), \\ \dot{\varphi}(t) &:= -b(t)\Lambda W_{\underline{\lambda(t)}}, \end{aligned}$$

where

$$(2.8) \quad \begin{aligned} b(t) &:= e^{-\kappa|T|} + \kappa^2 \int_T^t \frac{\lambda(\tau)}{\mu(\tau)^2} d\tau, & \text{for } t \in [T, T_0], \\ S(t) &:= \chi \cdot \left(\frac{\lambda(t)^2}{\mu(t)^2} P_{\lambda(t)} + b(t)^2 Q_{\lambda(t)} \right), & \text{for } t \in [T, T_0]. \end{aligned}$$

From (2.7) we get $\left(\frac{8}{9}\right)^3 \frac{1}{\kappa} e^{-\kappa|t|} \leq \frac{\lambda(t)}{\mu(t)^2} \leq \left(\frac{9}{8}\right)^3 \frac{1}{\kappa} e^{-\kappa|t|}$. Integrating we get the following bound for $b(t)$, $t \in [T, T_0]$:

$$(2.9) \quad \begin{aligned} \left(\frac{8}{9}\right)^3 e^{-\kappa|t|} &< e^{-\kappa|T|} + \left(\frac{8}{9}\right)^3 (e^{-\kappa|t|} - e^{-\kappa|T|}) \\ &\leq b(t) \leq e^{-\kappa|T|} + \left(\frac{9}{8}\right)^3 (e^{-\kappa|t|} - e^{-\kappa|T|}) < \left(\frac{9}{8}\right)^3 e^{-\kappa|t|}. \end{aligned}$$

From (2.4) we obtain

$$(2.10) \quad \begin{aligned} \|\chi \cdot P_{\underline{\lambda}}\|_{\dot{H}^1} &\simeq \|\partial_r(\chi \cdot P_{\underline{\lambda}})\|_{L^2(r^5 dr)} \lesssim \|P_{\underline{\lambda}}\|_{L^2(0 \leq r \leq 2)} + \left\| \frac{1}{\lambda} (\partial_r P)_{\underline{\lambda}} \right\|_{L^2(0 \leq r \leq 2)} \\ &= \|P\|_{L^2(0 \leq r \leq \frac{2}{\lambda})} + \frac{1}{\lambda} \|\partial_r P\|_{L^2(0 \leq r \leq \frac{2}{\lambda})} \\ &\lesssim \left(\int_0^{2/\lambda} (1+r^2)^{-2} r^5 dr \right)^{\frac{1}{2}} + \frac{1}{\lambda} \cdot \left(\int_0^{2/\lambda} (1+r^3)^{-2} r^5 dr \right)^{\frac{1}{2}} \lesssim \frac{1}{\lambda} |\log \lambda|^{\frac{1}{2}}, \end{aligned}$$

and analogously $\|\chi \cdot Q_{\underline{\lambda}}\|_{\dot{H}^1} \lesssim \frac{1}{\lambda} |\log \lambda|^{\frac{1}{2}}$, hence

$$\|S(t)\|_{\dot{H}^1} \leq \frac{\lambda^3}{\mu^2} \|\chi P_{\underline{\lambda}}\|_{\dot{H}^1} + \lambda b^2 \|\chi Q_{\underline{\lambda}}\|_{\dot{H}^1} \lesssim e^{-3\kappa|t|} \cdot \frac{1}{\lambda} |\log \lambda|^{\frac{1}{2}} \lesssim \sqrt{|t|} \cdot e^{-2\kappa|t|}.$$

Thus for any $c > 0$ there exists T_0 such that if $T < T_0$ then

$$(2.11) \quad \|S(t)\|_{\dot{H}^1} \leq c \cdot e^{-\frac{3}{2}\kappa|t|}, \quad \text{for } t \in [T, T_0].$$

Note also that $\|P_{\lambda}\|_{L^\infty} + \|Q_{\lambda}\|_{L^\infty} \lesssim \lambda^{-2}$, hence $S(t)$ is bounded in L^∞ .

Since \mathcal{Z} has compact support, for sufficiently small λ , (2.3) implies

$$(2.12) \quad \langle \underline{\mathcal{Z}_{\lambda(t)}}, S(t) \rangle = 0.$$

We denote

$$(2.13) \quad \begin{aligned} \psi(t) &= (\psi(t), \dot{\psi}(t)) := \partial_t \varphi(t) - DE(\varphi(t)) \\ &= (\partial_t \varphi(t) - \dot{\varphi}(t), \partial_t \dot{\varphi}(t) - (\Delta \varphi(t) + f(\varphi(t)))). \end{aligned}$$

This function describes how much $\varphi(t)$ fails to be an exact solution of (1.1). Before we prove bounds on $\psi(t)$, we gather in the next elementary lemma pointwise inequalities used in various places in the text.

Lemma 2.3. *Let $k, l, m \in \mathbb{R}$. Then*

$$(2.14) \quad |f'(k+l) - f'(k)| \leq f'(l),$$

$$(2.15) \quad |f(k+l) - f(k) - f'(k)l| \leq 5|f(l)|,$$

$$(2.16) \quad |F(k+l) - F(k) - f(k)l - \frac{1}{2}f'(k)l^2| \leq 5F(l).$$

Proof. Inequality (2.14) is well-known. Bounds (2.15) holds for $k = 0$, hence (by homogeneity) we may assume that $k = 1$. For $|l| \leq 1$ we have $|f(1+l) - f(1) - f'(1)l| = |(1+l)^2 - 1 - 2l| = 2l^2 \leq 5|f(l)|$ and for $|l| \geq 1$ we find $|f(1+l) - f(1) - f'(1)l| \leq (1+l)^2 + 1 + 2|l| \leq 5|f(l)|$. Bound (2.16) follows by integrating (2.15). \square

Lemma 2.4. *Suppose that for $t \in [T, T_0]$ there holds $|\lambda'(t)| \lesssim e^{-\kappa|t|}$ and $|\mu'(t)| \lesssim e^{-\kappa|t|}$. Then*

$$(2.17) \quad \|\psi(t) + \mu'(t) \frac{1}{\mu(t)} \Lambda W_{\mu(t)} + (\lambda'(t) - b(t)) \frac{1}{\lambda(t)} \Lambda W_{\lambda(t)}\|_{\dot{H}^1} \lesssim e^{-\frac{3}{2}\kappa|t|},$$

$$(2.18) \quad \|\dot{\psi}(t) - \frac{b(t)}{\lambda(t)} (\lambda'(t) - b(t)) \Lambda_0 \Lambda W_{\lambda(t)}\|_{L^2} \lesssim e^{-\frac{3}{2}\kappa|t|},$$

$$(2.19) \quad \|(-\Delta - f'(\varphi(t)))\psi(t)\|_{\dot{H}^{-1}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Proof. Using the definitions of φ and ψ we find

$$\begin{aligned} \psi + \mu' \Lambda W_{\underline{\mu}} + (\lambda' - b) \Lambda W_{\underline{\lambda}} &= \partial_t \varphi - \dot{\varphi} + \mu' \Lambda W_{\underline{\mu}} + (\lambda' - b) \Lambda W_{\underline{\lambda}} \\ &= -\mu' \Lambda W_{\underline{\mu}} - \lambda' \Lambda W_{\underline{\lambda}} + \partial_t S + b \Lambda W_{\underline{\lambda}} + (\lambda' - b) \Lambda W_{\underline{\lambda}} \\ &= \chi \cdot \left(-2\mu' \frac{\lambda^3}{\mu^3} P_{\underline{\lambda}} + 2\lambda' \frac{\lambda^2}{\mu^2} P_{\underline{\lambda}} - \lambda' \frac{\lambda^2}{\mu^2} \Lambda P_{\underline{\lambda}} + 2b' b \lambda Q_{\underline{\lambda}} - \lambda' b^2 \Lambda Q_{\underline{\lambda}} \right). \end{aligned}$$

Since ΛP and ΛQ are rational functions decaying like r^{-2} , we have $\|\chi \cdot \Lambda P_{\underline{\lambda}}\|_{\dot{H}^1} \lesssim \sqrt{|t|} \cdot e^{\kappa|t|}$ and $\|\chi \cdot \Lambda Q_{\underline{\lambda}}\|_{\dot{H}^1} \lesssim \sqrt{|t|} \cdot e^{\kappa|t|}$, see (2.10). This implies (2.17) because $|\lambda|, |b|, |\lambda'|, |b'|, |\mu'| \lesssim e^{-\kappa|t|}$.

In order to prove (2.18), we consider separately the regions $|x| \leq \sqrt{\lambda}$ and $|x| \geq \sqrt{\lambda}$. The first step is to treat the nonlinearity, that is to show that

$$(2.20) \quad \begin{aligned} &\|f(\varphi) - f(W_{\mu}) - f(W_{\lambda}) - f'(W_{\lambda})W_{\mu} \\ &\quad - \frac{\lambda^2}{\mu^2} f'(W_{\lambda})P_{\lambda} - b^2 f'(W_{\lambda})Q_{\lambda}\|_{L^2(|x| \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

Applying (2.15) with $k = W_{\lambda}$ and $l = W_{\mu} + S$ we get

$$|f(\varphi) - f(W_{\lambda}) - f'(W_{\lambda})(W_{\mu} + \lambda^2 \mu^{-2} \cdot P_{\lambda} + b^2 \cdot Q_{\lambda})| \lesssim |f(W_{\mu})| + |f(\lambda^2 \mu^{-2} P_{\lambda} + b^2 Q_{\lambda})|,$$

which is bounded in L^∞ , hence bounded by $\lambda^{\frac{3}{2}} \sim e^{-\frac{3}{2}\kappa|t|}$ in $L^2(|x| \leq \sqrt{\lambda})$. This proves (2.20). Now we check that

$$(2.21) \quad \|f'(W_{\lambda})W_{\mu} - \frac{1}{\mu^2} f'(W_{\lambda})\|_{L^2} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Indeed, for $|x| \leq \sqrt{\lambda}$ we have $|W_\mu(x) - \frac{1}{\mu^2}| = |W_\mu(x) - W_\mu(0)| \lesssim |x|^2 \leq \lambda \sim e^{-\kappa|t|}$, hence

$$\|f'(W_\lambda)W_\mu - \frac{1}{\mu^2}f'(W_\lambda)\|_{L^2(|x| \leq \sqrt{\lambda})} \lesssim \|W_\mu - \frac{1}{\mu^2}\|_{L^\infty(|x| \leq \sqrt{\lambda})} \cdot \|W_\lambda\|_{L^2} \lesssim e^{-\kappa|t|} \cdot \lambda \ll e^{-\frac{3}{2}\kappa|t|}.$$

From (2.20) and (2.21) we obtain

$$(2.22) \quad \|f(\varphi) - f(W_\mu) - f(W_\lambda) - \mu^{-2}f'(W_\lambda) - \lambda^2\mu^{-2}f'(W_\lambda)P_\lambda - b^2f'(W_\lambda)Q_\lambda\|_{L^2} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Since $\chi = 1$ in the region $|x| \leq \sqrt{\lambda}$, we have $\Delta\varphi = \Delta(W_\mu) + \Delta(W_\lambda) + \lambda^2\mu^{-2}\Delta(P_\lambda) + b^2\Delta(Q_\lambda)$. From this and (2.22), using the fact that $\Delta(W_\mu) + f(W_\mu) = \Delta(W_\lambda) + f(W_\lambda) = 0$, we get

$$(2.23) \quad \begin{aligned} & \|\Delta\varphi + f(\varphi) - \mu^{-2}(\lambda^2\Delta(P_\lambda) + \lambda^2f'(W_\lambda)P_\lambda + f'(W_\lambda)) \\ & - (b^2\Delta(Q_\lambda) + b^2f'(W_\lambda)Q_\lambda)\|_{L^2(|x| \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

But formula (2.2) gives

$$\begin{aligned} \lambda^2\Delta(P_\lambda) + \lambda^2f'(W_\lambda)P_\lambda &= (-LP)_\lambda = -\kappa^2\Lambda W_\lambda - f'(W_\lambda) = -\kappa^2\Lambda W_\lambda - f'(W_\lambda), \\ b^2\Delta(Q_\lambda) + b^2f'(W_\lambda)Q_\lambda &= \frac{b^2}{\lambda^2}(-LQ)_\lambda = \frac{b^2}{\lambda}\Lambda_0\Lambda W_\lambda, \end{aligned}$$

hence we can rewrite (2.23) as

$$(2.24) \quad \|\Delta\varphi + f(\varphi) + \frac{\kappa^2\lambda}{\mu^2}\Lambda W_\lambda - \frac{b^2}{\lambda}\Lambda_0\Lambda W_\lambda\|_{L^2(|x| \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

We have $\partial_t\varphi = -b'\Lambda W_\lambda + \frac{\lambda'b}{\lambda}\Lambda_0\Lambda W_\lambda$, thus

$$(2.25) \quad \begin{aligned} -\dot{\varphi} + \frac{b}{\lambda}(\lambda' - b)\Lambda_0\Lambda W_\lambda &= \Delta\varphi + f(\varphi) + b'\Lambda W_\lambda - \frac{b\lambda'}{\lambda}\Lambda_0\Lambda W_\lambda + \frac{b}{\lambda}(\lambda' - b)\Lambda_0\Lambda W_\lambda \\ &= \Delta\varphi + f(\varphi) + \frac{\kappa^2\lambda}{\mu^2}\Lambda W_\lambda - \frac{b^2}{\lambda}\Lambda_0\Lambda W_\lambda, \end{aligned}$$

so (2.24) yields (2.18) in the region $|x| \leq \sqrt{\lambda}$.

Consider now the region $|x| \geq \sqrt{\lambda}$. First we show that

$$(2.26) \quad \|\Delta\varphi - \Delta(W_\mu)\|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

To this end, we compute

$$\begin{aligned} \|\Delta(W_\lambda)\|_{L^2(|x| \geq \sqrt{\lambda})} &= \|f(W_\lambda)\|_{L^2(|x| \geq \sqrt{\lambda})} = \frac{1}{\lambda}\|f(W)\|_{L^2(|x| \geq 1/\sqrt{\lambda})} \\ &\lesssim \frac{1}{\lambda} \left(\int_{1/\sqrt{\lambda}}^{+\infty} r^{-16} r^5 dr \right)^{\frac{1}{2}} \lesssim \frac{1}{\lambda} \lambda^{\frac{5}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

We need to show that $\|\Delta(\chi \cdot P_\lambda)\|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim e^{\frac{1}{2}\kappa|t|}$ and $\|\Delta(\chi \cdot Q_\lambda)\|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim e^{\frac{1}{2}\kappa|t|}$. We will prove the first bound (the second is exactly the same). Notice that $|P_\lambda(x)| \lesssim \frac{1}{\lambda^2} \cdot \frac{\lambda^2}{|x|^2} = |x|^{-2}$ and similarly $|\nabla(P_\lambda)(x)| \lesssim |x|^{-3}$, $|\nabla^2(P_\lambda)(x)| \lesssim |x|^{-4}$, hence we have a pointwise bound

$$|\Delta(\chi P_\lambda)| \lesssim |\nabla^2\chi| \cdot |P_\lambda| + |\nabla\chi| \cdot |\nabla(P_\lambda)| + |\chi| \cdot |\nabla^2(P_\lambda)| \lesssim |\nabla^2\chi| \cdot |x|^{-2} + |\nabla\chi| \cdot |x|^{-3} + |\chi| \cdot |x|^{-4}.$$

Of course $\| |\nabla^2\chi| \cdot |x|^{-2} \|_{L^2(|x| \geq \sqrt{\lambda})} + \| |\nabla\chi| \cdot |x|^{-3} \|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim 1 \ll e^{\frac{1}{2}\kappa|t|}$ and we are left with the last term. We compute

$$\| |\chi| \cdot |x|^{-4} \|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim \left(\int_{\sqrt{\lambda}}^2 r^{-8} r^5 dr \right)^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \lesssim e^{\frac{1}{2}\kappa|t|}.$$

This finishes the proof of (2.26).

Applying (2.15) with $k = W_\mu$ and $l = W_\lambda + S$ we get

$$|f(\varphi) - f(W_\mu)| \lesssim f'(W_\mu) \cdot (|W_\lambda| + |S|) + |f(W_\lambda)| + |f(S)| \lesssim |W_\lambda| + |S|,$$

where the last estimate follows from the fact that $\|W_\lambda\|_{L^\infty} + \|S\|_{L^\infty} \lesssim 1$ for $|x| \geq \sqrt{\lambda}$.

We have $\|\chi \cdot P_\Delta\|_{L^2} \lesssim \left(\int_0^{\frac{2}{\lambda}} (r^{-2})^2 r^5 dr \right)^{\frac{1}{2}} \sim \lambda^{-1}$, and similarly $\|\chi \cdot Q_\Delta\|_{L^2} \lesssim \lambda^{-1}$, which implies $\|S\|_{L^2} \ll e^{-\frac{3}{2}\kappa|t|}$. There holds also

$$(2.27) \quad \|W_\lambda\|_{L^2(|x| \geq \sqrt{\lambda})} = \lambda \|W\|_{L^2(|x| \geq 1/\sqrt{\lambda})} \lesssim \lambda \left(\int_{1/\sqrt{\lambda}}^\infty r^{-8} r^5 dr \right)^{\frac{1}{2}} \sim \lambda^{\frac{3}{2}} \sim e^{-\frac{3}{2}\kappa|t|},$$

hence $\|f(\varphi) - f(W_\mu)\|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}$. Together with (2.26) this yields

$$\|\Delta\varphi + f(\varphi)\|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

The same computation as in (2.27) gives

$$\|\Lambda W_\Delta\|_{L^2(|x| \geq \sqrt{\lambda})} + \|\Lambda_0 \Lambda W_\Delta\|_{L^2(|x| \geq \sqrt{\lambda})} \lesssim e^{-\frac{1}{2}\kappa|t|},$$

hence (2.25) implies that (2.18) holds also in the region $|x| \geq \sqrt{\lambda}$.

We are left with (2.19). From (2.17) it follows that it suffices to check that

$$(2.28) \quad \left\| (-\Delta - f'(\varphi))(\lambda' - b) \frac{1}{\lambda} \Lambda W_\lambda \right\|_{\dot{H}^{-1}} \lesssim e^{-\frac{3}{2}\kappa|t|}$$

and

$$(2.29) \quad \left\| (-\Delta - f'(\varphi))\mu' \frac{1}{\mu} \Lambda W_\mu \right\|_{\dot{H}^{-1}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

We start with (2.28). Since $|\lambda' - b| \lesssim e^{-\kappa|t|} \lesssim \lambda$, we need to show that

$$(2.30) \quad \|(-\Delta - f'(\varphi))\Lambda W_\lambda\|_{\dot{H}^{-1}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

By Hölder inequality

$$(2.31) \quad \|f'(W_\mu)\Lambda W_\lambda\|_{L^{\frac{3}{2}}} \leq \|f'(W_\mu)\|_{L^{12}} \cdot \|\Lambda W_\lambda\|_{L^{\frac{12}{7}}} \lesssim \lambda^{\frac{3}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Since $|f'(W_\lambda + W_\mu) - f'(W_\lambda)| \leq f'(W_\mu)$, we obtain

$$\|(f'(W_\lambda + W_\mu) - f'(W_\lambda))\Lambda W_\lambda\|_{\dot{H}^{-1}} \lesssim \|(f'(W_\lambda + W_\mu) - f'(W_\lambda))\Lambda W_\lambda\|_{L^{\frac{3}{2}}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

As noted earlier $(-\Delta - f'(W_\lambda))\Lambda W_\lambda = 0$, hence

$$\|(-\Delta - f'(W_\lambda + W_\mu))\Lambda W_\lambda\|_{\dot{H}^{-1}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

From (2.11) we have $\|f'(\varphi) - f'(W_\lambda + W_\mu)\|_{L^3} \lesssim e^{-\frac{3}{2}\kappa|t|}$. This implies (2.30).

The proof of (2.29) is similar. It suffices to check that $\|f'(W_\lambda)\Lambda W_\mu\|_{\dot{H}^{-1}} \lesssim e^{-\frac{1}{2}\kappa|t|}$, and in fact we even have the bound $\lesssim e^{-\frac{3}{2}\kappa|t|}$, with the same proof as in (2.31). \square

3. BOOTSTRAP CONTROL OF THE ERROR TERM – THE NLW CASE

In the preceding section we defined *approximate* solutions of (1.1). In the present section we consider *exact* solutions of (1.1), with some specific initial data prescribed at $t = T$, with $T \rightarrow -\infty$. Our goal is to control the evolution of this solution up to a time T_0 independent of T .

For technical reasons we will require the initial data to belong to the space $X^1 \times H^1$, where $X^1 := \dot{H}^2 \cap \dot{H}^1$. This regularity is preserved by the flow, see Proposition A.1.

3.1. Set-up of the bootstrap. It is known that $L = -\Delta - f'(W)$ has exactly one strictly negative simple eigenvalue which we denote $-\nu^2$ (we take $\nu > 0$). We denote the corresponding positive eigenfunction \mathcal{Y} , normalized so that $\|\mathcal{Y}\|_{L^2} = 1$. By elliptic regularity \mathcal{Y} is smooth and by Agmon estimates it decays exponentially. Self-adjointness of L implies that

$$\langle \mathcal{Y}, \Lambda W \rangle = 0.$$

Note that

$$(3.1) \quad \nu < 1.$$

Indeed, it is well-known that $-\Delta - W \geq 0$, with a one-dimensional kernel generated by W . Since $1 - W(x) > 0$ almost everywhere, for any $h \neq 0$ we have

$$\langle h, Lh \rangle + \langle h, h \rangle = \langle (-\Delta - 2W + 1)h, h \rangle > \langle (-\Delta - W)h, h \rangle \geq 0.$$

We define

$$\mathcal{Y}^- := \left(\frac{1}{\nu}\mathcal{Y}, -\mathcal{Y}\right), \quad \mathcal{Y}^+ := \left(\frac{1}{\nu}\mathcal{Y}, \mathcal{Y}\right),$$

$$\alpha^- := \frac{\nu}{2}J\mathcal{Y}^+ = \frac{1}{2}(\nu\mathcal{Y}, -\mathcal{Y}), \quad \alpha^+ := -\frac{\nu}{2}J\mathcal{Y}^- = \frac{1}{2}(\nu\mathcal{Y}, \mathcal{Y}).$$

We have $J \circ D^2E(\mathbf{W}) = \begin{pmatrix} 0 & \text{Id} \\ -L & 0 \end{pmatrix}$. A short computation shows that

$$J \circ D^2E(\mathbf{W})\mathcal{Y}^- = -\nu\mathcal{Y}^-, \quad J \circ D^2E(\mathbf{W})\mathcal{Y}^+ = \nu\mathcal{Y}^+$$

and

$$\langle \alpha^-, J \circ D^2E(\mathbf{W})\mathbf{h} \rangle = -\nu\langle \alpha^-, \mathbf{h} \rangle, \quad \langle \alpha^+, J \circ D^2E(\mathbf{W})\mathbf{h} \rangle = \nu\langle \alpha^+, \mathbf{h} \rangle, \quad \forall \mathbf{h} \in \mathcal{E}.$$

We will think of α^- and α^+ as linear forms on \mathcal{E} . Notice that $\langle \alpha^-, \mathcal{Y}^- \rangle = \langle \alpha^+, \mathcal{Y}^+ \rangle = 1$ and $\langle \alpha^-, \mathcal{Y}^+ \rangle = \langle \alpha^+, \mathcal{Y}^- \rangle = 0$.

The rescaled versions of these objects are

$$\mathcal{Y}_\lambda^- := \left(\frac{1}{\nu}\mathcal{Y}_\lambda, -\mathcal{Y}_\lambda\right), \quad \mathcal{Y}_\lambda^+ := \left(\frac{1}{\nu}\mathcal{Y}_\lambda, \mathcal{Y}_\lambda\right),$$

$$(3.2) \quad \alpha_\lambda^- := \frac{\nu}{2\lambda}J\mathcal{Y}_\lambda^+ = \frac{1}{2}\left(\frac{\nu}{\lambda}\mathcal{Y}_\lambda, -\mathcal{Y}_\lambda\right), \quad \alpha_\lambda^+ := -\frac{\nu}{2\lambda}J\mathcal{Y}_\lambda^- = \frac{1}{2}\left(\frac{\nu}{\lambda}\mathcal{Y}_\lambda, \mathcal{Y}_\lambda\right).$$

The scaling is chosen so that $\langle \alpha_\lambda^-, \mathcal{Y}_\lambda^- \rangle = \langle \alpha_\lambda^+, \mathcal{Y}_\lambda^+ \rangle = 1$. We have

$$(3.3) \quad J \circ D^2E(\mathbf{W}_\lambda)\mathcal{Y}_\lambda^- = -\frac{\nu}{\lambda}\mathcal{Y}_\lambda^-, \quad J \circ D^2E(\mathbf{W}_\lambda)\mathcal{Y}_\lambda^+ = \frac{\nu}{\lambda}\mathcal{Y}_\lambda^+$$

and

$$(3.4) \quad \langle \alpha_\lambda^-, J \circ D^2E(\mathbf{W}_\lambda)\mathbf{h} \rangle = -\frac{\nu}{\lambda}\langle \alpha_\lambda^-, \mathbf{h} \rangle, \quad \langle \alpha_\lambda^+, J \circ D^2E(\mathbf{W}_\lambda)\mathbf{h} \rangle = \frac{\nu}{\lambda}\langle \alpha_\lambda^+, \mathbf{h} \rangle, \quad \forall \mathbf{h} \in \mathcal{E}.$$

We will need the following simple lemma in order to properly choose the initial data.

Lemma 3.1. *There exist universal constants $\eta, C > 0$ such that if $0 < \lambda < \eta \cdot \mu$, then for all $a_0 \in \mathbb{R}$ there exists $\mathbf{h}_0 \in X^1 \times H^1$ satisfying the orthogonality conditions $\langle \mathcal{Z}_\mu, \mathbf{h}_0 \rangle = \langle \mathcal{Z}_\lambda, \mathbf{h}_0 \rangle = 0$ and such that $\langle \alpha_\mu^+, \mathbf{h}_0 \rangle = 0$, $\langle \alpha_\mu^-, \mathbf{h}_0 \rangle = 0$, $\langle \alpha_\lambda^+, \mathbf{h}_0 \rangle = a_0$, $\langle \alpha_\lambda^-, \mathbf{h}_0 \rangle = 0$, $\|\mathbf{h}_0\|_{\mathcal{E}} \leq C|a_0|$.*

Proof. We consider functions of the form:

$$\mathbf{h}_0 := a_2^+ \mathcal{Y}_\mu^+ + a_2^- \mathcal{Y}_\mu^- + b_2 \Lambda \mathbf{W}_\mu + a_1^+ \mathcal{Y}_\lambda^+ + a_1^- \mathcal{Y}_\lambda^- + b_1 \Lambda \mathbf{W}_\lambda, \quad a_2^\pm, a_1^\pm, b_2, b_1 \in \mathbb{R}.$$

Consider the linear map $\Phi : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ defined as follows:

$$\Phi(a_2^+, a_2^-, b_2, a_1^+, a_1^-, b_1) := (\langle \alpha_\mu^+, \mathbf{h}_0 \rangle, \langle \alpha_\mu^-, \mathbf{h}_0 \rangle, \langle \frac{1}{\mu} \mathcal{Z}_\mu, h_0 \rangle, \langle \alpha_\lambda^+, \mathbf{h}_0 \rangle, \langle \alpha_\lambda^-, \mathbf{h}_0 \rangle, \langle \frac{1}{\lambda} \mathcal{Z}_\lambda, h_0 \rangle)$$

It is easy to check that the matrix of Φ is strictly diagonally dominant if η is small enough. \square

We consider the solution $\mathbf{u}(t) = \mathbf{u}(a_0; t) : [T, T_+) \rightarrow \mathcal{E}$ of (1.1) with the initial data

$$(3.5) \quad \mathbf{u}(T) = (W_{\frac{1}{\kappa}e^{-\kappa|T|}} + W + h_0, -\Lambda W_{\frac{1}{\kappa}e^{-\kappa|T|}}),$$

where h_0 is the function given by Lemma 3.1 with $\lambda = \frac{1}{\kappa}e^{-\kappa|T|}$, $\mu = 1$ and some a_0 chosen later, satisfying

$$|a_0| \leq e^{-\frac{3}{2}\kappa|T|}.$$

Note that the initial data depend continuously on a_0 .

For $t \geq T$ we define the functions $\lambda(t)$ and $\mu(t)$ as the solutions of the following system of ordinary differential equations with the initial data $\mu(T) = 1$ and $\lambda(T) = \frac{1}{\kappa}e^{-\kappa|T|}$:

$$(3.6) \quad \begin{pmatrix} \langle \mathcal{Z}_\lambda, \Lambda W_\lambda \rangle - \langle \frac{1}{\lambda} \Lambda_0 \mathcal{Z}_\lambda, h \rangle & \langle \mathcal{Z}_\lambda, \Lambda W_\mu \rangle \\ \langle \mathcal{Z}_\mu, \Lambda W_\lambda \rangle & \langle \mathcal{Z}_\mu, \Lambda W_\mu \rangle - \langle \frac{1}{\mu} \Lambda_0 \mathcal{Z}_\mu, h \rangle \end{pmatrix} \cdot \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \begin{pmatrix} -\langle \mathcal{Z}_\lambda, \partial_t u \rangle \\ -\langle \mathcal{Z}_\mu, \partial_t u \rangle \end{pmatrix},$$

where

$$h = h(t) := \begin{cases} u(t) - W_{\mu(t)} - W_{\lambda(t)}, & \text{if } \|u(t) - W_{\mu(t)} - W_{\lambda(t)}\|_{\dot{H}^1} \leq \eta, \\ \frac{\eta}{\|u(t) - W_{\mu(t)} - W_{\lambda(t)}\|_{\dot{H}^1}} (u(t) - W_{\mu(t)} - W_{\lambda(t)}), & \text{if } \|u(t) - W_{\mu(t)} - W_{\lambda(t)}\|_{\dot{H}^1} \geq \eta \end{cases}$$

with a small constant $\eta > 0$. Notice that $\langle \mathcal{Z}_\lambda, \Lambda W_\lambda \rangle = \langle \mathcal{Z}_\mu, \Lambda W_\mu \rangle = \langle \mathcal{Z}, \Lambda W \rangle > 0$, $|\langle \frac{1}{\lambda} \Lambda_0 \mathcal{Z}_\lambda, h \rangle| + |\langle \frac{1}{\mu} \Lambda_0 \mathcal{Z}_\mu, h \rangle| \lesssim \|h\|_{\dot{H}^1}$ and $|\langle \mathcal{Z}_\lambda, \Lambda W_\mu \rangle| + |\langle \mathcal{Z}_\mu, \Lambda W_\lambda \rangle| \lesssim \lambda/\mu$. For $t < T_0$ bounds (2.7) imply that λ/μ is small, hence equation (3.6) defines a unique solution as long as (2.7) holds.

Remark 3.2. Actually the second case in the definition of $h(t)$ will never occur in our analysis, since the bootstrap assumptions imply that $\|h(t)\|_{\dot{H}^1}$ is small.

Suppose that $\lambda(t)$ and $\mu(t)$ are well defined and satisfy (2.7) for $t \in [T, T_1]$, where $T < T_1 \leq T_0$. Suppose also that $\|h(t)\|_{\dot{H}^1} < \eta$ for $t \in [T, T_1]$, which implies that $h(t) = u(t)$. Using (3.6) we find $\frac{d}{dt} \langle \mathcal{Z}_{\mu(t)}, h(t) \rangle = 0$ and $\frac{d}{dt} \langle \mathcal{Z}_{\lambda(t)}, h(t) \rangle = 0$. Since $\langle \mathcal{Z}_{\mu(T)}, h(T) \rangle = 0$ and $\langle \mathcal{Z}_{\lambda(T)}, h(T) \rangle = 0$, we obtain

$$(3.7) \quad \langle \mathcal{Z}_{\mu(t)}, h(t) \rangle = \langle \mathcal{Z}_{\lambda(t)}, h(t) \rangle = 0, \quad \text{for } t \in [T, T_1].$$

We denote $\mathbf{h}(t) := (h(t), \partial_t u(t))$, so that

$$(3.8) \quad \begin{cases} \partial_t h = \dot{h} + \mu' \Lambda W_\mu + \lambda' \Lambda W_\lambda, \\ \partial_t \dot{h} = \Delta h + f(W_\mu + W_\lambda + h) - f(W_\mu) - f(W_\lambda). \end{cases}$$

We define the function $b(t) : [T, T_1] \rightarrow \mathbb{R}$ by formula (2.8) and decompose

$$\mathbf{u}(t) = \boldsymbol{\varphi}(t) + \mathbf{g}(t), \quad t \in [T, T_1].$$

By the definitions of $\mathbf{g}(t)$ and $\boldsymbol{\psi}(t)$, $\mathbf{g}(t)$ satisfies the differential equation

$$(3.9) \quad \partial_t \mathbf{g}(t) = J \circ DE(\boldsymbol{\varphi}(t) + \mathbf{g}(t)) - J \circ DE(\boldsymbol{\varphi}(t)) - \boldsymbol{\psi}(t).$$

Finally, we denote

$$\begin{aligned} a_1^+(t) &:= \langle \alpha_{\lambda(t)}^+, \mathbf{g}(t) \rangle, & a_1^-(t) &:= \langle \alpha_{\lambda(t)}^-, \mathbf{g}(t) \rangle, \\ a_2^+(t) &:= \langle \alpha_{\mu(t)}^+, \mathbf{g}(t) \rangle, & a_2^-(t) &:= \langle \alpha_{\mu(t)}^-, \mathbf{g}(t) \rangle. \end{aligned}$$

The rest of this section is devoted to the proof of the following bootstrap estimate, which is the heart of the whole construction.

Proposition 3.3. *There exist constants $C_0 > 0$ and $T_0 < 0$ (C_0 and $|T_0|$ large) with the following property. Let $T < T_1 < T_0$ and suppose that $\mathbf{u}(t) = \boldsymbol{\varphi}(t) + \mathbf{g}(t) \in C([T, T_1]; X^1 \times H^1)$ is a solution of (1.1) with initial data (3.5) such that for $t \in [T, T_1]$ condition (2.7) is satisfied and*

$$(3.10) \quad \|\mathbf{g}(t)\|_{\mathcal{E}} \leq C_0 \cdot e^{-\frac{3}{2}\kappa|t|},$$

$$(3.11) \quad |a_1^+(t)| \leq e^{-\frac{3}{2}\kappa|t|}.$$

Then for $t \in [T, T_1]$ there holds

$$(3.12) \quad \|\mathbf{g}(t)\|_{\mathcal{E}} \leq \frac{1}{2}C_0 e^{-\frac{3}{2}\kappa|t|},$$

$$(3.13) \quad \left| \lambda(t) - \frac{1}{\kappa} e^{-\kappa|t|} \right| + |\mu(t) - 1| \lesssim C_0 e^{-\frac{3}{2}\kappa|t|}.$$

Remark 3.4. Notice that (3.12) and (3.13) are strictly stronger than (3.10) and (2.7) respectively, which will be crucial for closing the bootstrap in Subsection 3.6.

Remark 3.5. The same conclusion should be true without the assumption of $X^1 \times H^1$ regularity, by means of a standard approximation procedure (both the assumptions and the conclusion are continuous for the topology $\|\cdot\|_{\mathcal{E}}$).

3.2. Modulation.

Lemma 3.6. *Under assumptions (2.7) and (3.10), for $t \in [T, T_1]$ there holds*

$$(3.14) \quad \left\langle \frac{1}{\lambda(t)} \mathcal{Z}_{\lambda(t)}, g(t) \right\rangle = 0, \quad \left| \left\langle \frac{1}{\mu(t)} \mathcal{Z}_{\mu(t)}, g(t) \right\rangle \right| \lesssim c \cdot e^{-\frac{3}{2}\kappa|t|},$$

$$(3.15) \quad |\lambda'(t) - b(t)| + |\mu'(t)| \lesssim \|\mathbf{g}(t)\|_{\mathcal{E}} + c \cdot e^{-\frac{3}{2}\kappa|t|},$$

with a constant c arbitrarily small.

Proof. We have $g(t) = h(t) - S(t)$. Since \mathcal{Z} has compact support, (2.12) and (3.7) yield $\langle \mathcal{Z}_{\lambda}, g \rangle = 0$. From $\|\frac{1}{\mu} \mathcal{Z}_{\mu}\|_{L^\infty} \lesssim 1$ and $\|\chi \cdot \lambda^2 P_{\lambda}\|_{L^1} \leq \|\lambda^2 P_{\lambda}\|_{L^1(|x| \leq 2)} \lesssim \lambda^2 \sim e^{-2\kappa|t|}$ (analogously $\|\chi \cdot b^2 Q_{\lambda}\|_{L^1} \lesssim e^{-2\kappa|t|}$) we obtain $|\langle \frac{1}{\mu} \mathcal{Z}_{\mu}, g \rangle| \lesssim e^{-2\kappa|t|}$.

From (2.11) we have

$$(3.16) \quad \|g(t) - h(t)\|_{\dot{H}^1} \leq c \cdot e^{-\frac{3}{2}\kappa|t|}, \quad \text{with a small constant } c,$$

Using this, (3.6) yields

$$\begin{aligned} & \begin{pmatrix} \langle \mathcal{Z}_{\lambda}, \Lambda W_{\lambda} \rangle - \langle \frac{1}{\lambda} \Lambda_0 \mathcal{Z}_{\lambda}, g \rangle & \langle \mathcal{Z}_{\lambda}, \Lambda W_{\mu} \rangle \\ \langle \mathcal{Z}_{\mu}, \Lambda W_{\lambda} \rangle & \langle \mathcal{Z}_{\mu}, \Lambda W_{\mu} \rangle - \langle \frac{1}{\mu} \Lambda_0 \mathcal{Z}_{\mu}, g \rangle \end{pmatrix} \cdot \begin{pmatrix} \lambda' - b \\ \mu' \end{pmatrix} \\ &= \begin{pmatrix} -\langle \mathcal{Z}_{\lambda}, \partial_t u + b \Lambda W_{\lambda} \rangle \\ -\langle \mathcal{Z}_{\mu}, \partial_t u + b \Lambda W_{\lambda} \rangle \end{pmatrix} + o_{\mathbb{R}^2}(e^{-\frac{3}{2}\kappa|t|}). \end{aligned}$$

Since by definition $\partial_t u + b \Lambda W_{\lambda} = \dot{g}$, inverting the matrix we get (3.15). \square

Lemma 3.7. *Under assumptions (2.7) and (3.10), for $t \in [T, T_1]$ the functions $a_1^\pm(t)$ and $a_2^\pm(t)$ satisfy*

$$(3.17) \quad \left| \frac{d}{dt} a_1^+(t) - \frac{\nu}{\lambda(t)} a_1^+(t) \right| \leq c \cdot e^{-\frac{1}{2}\kappa|t|}, \quad \text{with a small constant } c$$

$$(3.18) \quad |a_1^-(t)| \leq e^{-\frac{3}{2}\kappa|t|},$$

$$(3.19) \quad |a_2^\pm(t)| \leq e^{-\frac{3}{2}\kappa|t|}.$$

Proof. Using the definition of $a_1^+(t)$ we compute

$$(3.20) \quad \begin{aligned} \frac{d}{dt} a_1^+(t) &= \frac{d}{dt} \langle \alpha_{\lambda(t)}^+, \mathbf{g}(t) \rangle \\ &= \left\langle -\frac{\lambda'}{\lambda} (\Lambda_{\mathcal{E}^*} \alpha^+)_{\lambda}, \mathbf{g} \right\rangle + \langle \alpha_{\lambda}^+, J \circ DE(\boldsymbol{\varphi} + \mathbf{g}) - J \circ DE(\boldsymbol{\varphi}) - \boldsymbol{\psi} \rangle, \end{aligned}$$

where $\Lambda_{\mathcal{E}^*} \alpha^+ := -\frac{\partial}{\partial \lambda} \alpha_{\lambda}^+|_{\lambda=1}$. We have

$$(3.21) \quad |\langle (\Lambda_{\mathcal{E}^*} \alpha^+)_{\lambda}, \mathbf{g} \rangle| \lesssim \|\mathbf{g}\|_{\mathcal{E}} \ll e^{-\frac{1}{2}\kappa|t|}.$$

Since $\langle \alpha_{\lambda}^+, (\Lambda W_{\lambda}, 0) \rangle = 0$, using (2.18) we obtain

$$(3.22) \quad |\langle \alpha_{\lambda}^+, \boldsymbol{\psi} \rangle| \lesssim \left\| \boldsymbol{\psi} - \frac{\lambda' - b}{\lambda} (\Lambda W_{\lambda}, 0) \right\|_{\mathcal{E}} \ll e^{-\frac{1}{2}\kappa|t|}.$$

From (2.15) we obtain $\|f(\varphi + g) - f(\varphi) - f'(\varphi)g\|_{L^{\frac{3}{2}}} \lesssim \|g\|_{\dot{H}^1}^2$. From (2.14) and (2.11) we have $\|(f'(\varphi) - f'(W_{\mu}) - f'(W_{\lambda}))g\|_{L^{\frac{3}{2}}} \lesssim \|\varphi - W_{\mu} - W_{\lambda}\|_{L^3} \cdot \|g\|_{L^3} \lesssim e^{-\frac{3}{2}\kappa|t|} \|g\|_{\dot{H}^1}$. Taking the sum we obtain

$$(3.23) \quad \|f(\varphi + g) - f(\varphi) - (f'(W_{\mu}) + f'(W_{\lambda}))g\|_{L^{\frac{3}{2}}} \lesssim \|g\|_{\dot{H}^1}^2 + e^{-\frac{3}{2}\kappa|t|} \|g\|_{\dot{H}^1}.$$

But $|\langle \mathcal{Y}_{\underline{\lambda}}, f'(W_{\mu})g \rangle| \lesssim \|\mathcal{Y}_{\underline{\lambda}}\|_{L^{\frac{3}{2}}} \cdot \|f'(W_{\mu})\|_{L^{\infty}} \cdot \|g\|_{L^3} \lesssim \lambda \|g\|_{\dot{H}^1}$, hence

$$(3.24) \quad |\langle \mathcal{Y}_{\underline{\lambda}}, f(\varphi + g) - f(\varphi) - f'(W_{\lambda})g \rangle| \lesssim \frac{1}{\lambda} (\|g\|_{\dot{H}^1}^2 + e^{-\frac{3}{2}\kappa|t|} \|g\|_{\dot{H}^1}) \ll e^{-\frac{1}{2}\kappa|t|}.$$

Combining (3.20) with (3.21), (3.22) and (3.24) we obtain

$$\frac{d}{dt} a_1^+(t) = \langle \alpha_{\lambda}^+, J \circ D^2 E(\mathbf{W}_{\lambda}) \mathbf{g} \rangle + o(e^{-\frac{1}{2}\kappa|t|}) = \frac{\nu}{\lambda} a_1^+(t) + o(e^{-\frac{1}{2}\kappa|t|}),$$

where in the last step we use (3.4). This proves (3.17).

Similarly, we have

$$(3.25) \quad \left| \frac{d}{dt} a_1^-(t) + \frac{\nu}{\lambda(t)} a_1^-(t) \right| \leq c \cdot e^{-\frac{1}{2}\kappa|t|}.$$

Inequality (3.16) implies that (3.18) holds for t in a neighborhood of T . Suppose that $T_2 \in (T, T_1)$ is the last time such that (3.18) holds for $t \in [T, T_2]$. But (3.25) implies that $\frac{d}{dt} a_1^-(T_2)$ and $a_1^-(T_2)$ have opposite signs. Hence (3.18) cannot break down at $t = T_2$. The contradiction shows that (3.18) holds for $t \in [T, T_1]$.

In order to prove (3.19), it is more convenient to work with $\mathbf{h}(t)$, which was defined right before (3.8), than with $\mathbf{g}(t)$. We will prove the bound for $a_2^+(t)$. The proof for $a_2^-(t)$ is exactly the same. Let $\tilde{a}(t) := \langle \alpha_{\mu(t)}^+, \mathbf{h}(t) \rangle$. We have

$$(3.26) \quad \begin{aligned} |\langle \mathcal{Y}_{\underline{\mu}}, \Lambda W_{\underline{\lambda}} \rangle| &\lesssim \|\Lambda W_{\underline{\lambda}}\|_{L^1(|x| \leq 1)} + \|\Lambda W_{\underline{\lambda}}\|_{L^{\infty}(|x| \geq 1)} \\ &\lesssim \lambda^3 \int_0^{\frac{1}{\lambda}} r^{-4} r^5 dr + \frac{1}{\lambda^3} \cdot \left(\frac{1}{\lambda}\right)^{-4} \lesssim \lambda. \end{aligned}$$

Together with (2.11) this yields

$$|a_2^+(t) - \tilde{a}(t)| = |\langle \alpha_\mu^+, \mathbf{g}(t) - \mathbf{h}(t) \rangle| \lesssim b |\langle \mathcal{Y}_\mu, \Lambda W_\Delta \rangle| + \|S\|_{\dot{H}^1} \leq \frac{1}{2} e^{-\frac{3}{2}\kappa|t|},$$

hence it suffices to show that

$$(3.27) \quad |\tilde{a}(t)| \leq \frac{1}{2} e^{-\frac{3}{2}\kappa|t|}.$$

As in the case of $a_1^+(t)$, using (3.8) we obtain

$$(3.28) \quad \begin{aligned} \frac{d}{dt} \tilde{a}(t) &= \left\langle -\frac{\mu'}{\mu} (\Lambda \mathcal{E}^* \alpha^+)_\mu, \mathbf{h} \right\rangle + \langle \alpha_\mu^+, J \circ D^2 E(\mathbf{W}_\mu) h \rangle \\ &\quad + \langle \alpha_\mu^+, (\mu' \Lambda W_\mu + \lambda' \Lambda W_\Delta, f(W_\mu + W_\lambda + h) - f(W_\mu) - f(W_\lambda) - f'(W_\mu) h) \rangle \end{aligned}$$

But

$$\begin{aligned} \langle \mathcal{Y}_\mu, \Lambda W_\mu \rangle &= 0, \\ |\langle \mathcal{Y}_\mu, \Lambda W_\Delta \rangle| &\lesssim e^{-\kappa|t|}, \quad \text{see (3.26),} \\ |\langle \mathcal{Y}_\mu, f(W_\mu + W_\lambda + h) - f(W_\mu + W_\lambda) - f'(W_\mu + W_\lambda) h \rangle| &\lesssim \|h\|_{\dot{H}^1}^2 \lesssim e^{-2\kappa|t|}, \\ |\langle \mathcal{Y}_\mu, f(W_\mu + W_\lambda) - f(W_\mu) - f(W_\lambda) \rangle| &= 2 |\langle \mathcal{Y}_\mu, W_\mu \cdot W_\lambda \rangle| \\ &= 2 |\langle \mathcal{Y}_\mu \cdot W_\mu, W_\lambda \rangle| \lesssim e^{-2\kappa|t|}, \quad \text{see (3.26),} \\ |\langle \mathcal{Y}_\mu, (f'(W_\mu + W_\lambda) - f'(W_\mu)) h \rangle| &\lesssim \|\mathcal{Y}_\mu\|_{L^6} \cdot \|W_\lambda\|_{L^2} \cdot \|h\|_{L^3} \lesssim \lambda \|h\|_{\dot{H}^1} \lesssim e^{-2\kappa|t|}, \end{aligned}$$

hence (3.28) yields

$$\left| \frac{d}{dt} \tilde{a}(t) - \frac{\nu}{\mu(t)} \tilde{a}(t) \right| \lesssim e^{-2\kappa|t|} \leq c \cdot e^{-\frac{3}{2}\kappa|t|},$$

with a constant c arbitrarily small. Using (2.7), we get

$$(3.29) \quad \left| \frac{d}{dt} \tilde{a}(t) \right| \leq \frac{9\nu}{8} |\tilde{a}(t)| + c \cdot e^{-\frac{3}{2}\kappa|t|}.$$

As in the proof of (3.25), suppose that $T_2 \in (T, T_1)$ is the last time such that (3.27) holds for $t \in [T, T_2]$. This implies that $|\tilde{a}(T_2)| = \frac{1}{2} e^{-\frac{3}{2}\kappa|T_2|}$ and $|\frac{d}{dt} \tilde{a}(T_2)| \geq \frac{3}{4} \kappa \cdot e^{-\frac{3}{2}\kappa|T_2|}$, thus (3.29) yields

$$\frac{3}{4} \kappa \cdot e^{-\frac{3}{2}\kappa|T_2|} \leq \frac{9\nu}{16} e^{-\frac{3}{2}\kappa|T_2|} + c \cdot e^{-\frac{3}{2}\kappa|T_2|}.$$

But (2.1) and (3.1) give $\frac{3}{4} \kappa > \frac{9\nu}{16}$. Since c is arbitrarily small, we obtain a contradiction. \square

3.3. Coercivity. Recall that we denote $L := -\Delta - f'(W)$.

Lemma 3.8. *There exist constants $c, C > 0$ such that*

- *for all $g \in \dot{H}^1$ radially symmetric there holds*

$$\langle g, Lg \rangle = \int_{\mathbb{R}^6} |\nabla g|^2 dx - \int_{\mathbb{R}^N} f'(W) |g|^2 dx \geq c \int_{\mathbb{R}^6} |\nabla g|^2 dx - C(\langle \mathcal{Z}, g \rangle^2 + \langle \mathcal{Y}, g \rangle^2),$$

- *if $r_1 > 0$ is large enough, then for all $g \in \dot{H}_{\text{rad}}^1$ there holds*

$$(3.30) \quad (1 - 2c) \int_{|x| \leq r_1} |\nabla g|^2 dx + c \int_{|x| \geq r_1} |\nabla g|^2 dx - \int_{\mathbb{R}^6} f'(W) |g|^2 dx \geq -C(\langle \mathcal{Z}, g \rangle^2 + \langle \mathcal{Y}, g \rangle^2),$$

• if $r_2 > 0$ is small enough, then for all $g \in \dot{H}_{\text{rad}}^1$ there holds

$$(3.31) \quad (1 - 2c) \int_{|x| \geq r_2} |\nabla g|^2 dx + c \int_{|x| \leq r_2} |\nabla g|^2 dx - \int_{\mathbb{R}^6} f'(W) |g|^2 dx \geq -C(\langle \mathcal{Z}, g \rangle^2 + \langle \mathcal{Y}, g \rangle^2).$$

Proof. This is exactly Lemma 2.1 in [22], see also [30, Lemma 2.1]. \square

Lemma 3.9. *There exists a constant $\eta > 0$ such that if $\frac{\lambda}{\mu} < \eta$ and $\|U - (W_\mu + W_\lambda)\|_{\mathcal{E}} < \eta$, then for all $g \in \mathcal{E}$ there holds*

$$\frac{1}{2} \langle D^2 E(U) g, g \rangle + 2(\langle \alpha_\lambda^-, g \rangle^2 + \langle \alpha_\lambda^+, g \rangle^2 + \langle \frac{1}{\lambda} \mathcal{Z}_\Delta, g \rangle^2 + \langle \alpha_\mu^-, g \rangle^2 + \langle \alpha_\mu^+, g \rangle^2 + \langle \frac{1}{\mu} \mathcal{Z}_\mu, g \rangle^2) \gtrsim \|g\|_{\mathcal{E}}^2.$$

Proof. We will repeat with minor changes the proof of [22, Lemma 3.5].

Step 1. Without loss of generality we can assume that $\mu = 1$. Consider the operator L_λ defined by the following formula:

$$L_\lambda := \begin{pmatrix} -\Delta - f'(W_\lambda) - f'(W) & 0 \\ 0 & \text{Id} \end{pmatrix}.$$

From the fact that $\|f'(U) - f'(W) - f'(W_\lambda)\|_{L^3} \lesssim \|U - (W + W_\lambda)\|_{L^3}$ we obtain

$$(3.32) \quad |\langle D^2 E(U) g, g \rangle - \langle L_\lambda g, g \rangle| \leq c \|g\|_{\mathcal{E}}^2, \quad \forall g \in \mathcal{E},$$

with $c > 0$ small when η and λ_0 are small.

Step 2. In view of (3.32), it suffices to prove that if $\lambda < \lambda_0$, then

$$\frac{1}{2} \langle L_\lambda g, g \rangle + 2(\langle \alpha_{\lambda_1}^-, g \rangle^2 + \langle \alpha_{\lambda_1}^+, g \rangle^2 + \langle \alpha_{\lambda_2}^-, g \rangle^2 + \langle \alpha_{\lambda_2}^+, g \rangle^2 + \langle \frac{1}{\lambda} \mathcal{Z}_\Delta, g \rangle^2 + \langle \mathcal{Z}, g \rangle^2) \gtrsim \|g\|_{\mathcal{E}}^2.$$

Let $a_1^- := \langle \alpha_\lambda^-, g \rangle$, $a_1^+ := \langle \alpha_\lambda^+, g \rangle$, $a_2^- := \langle \alpha^- , g \rangle$, $a_2^+ := \langle \alpha^+ , g \rangle$, $b_1 := \langle \mathcal{Z}, \Lambda W \rangle^{-1} \cdot \langle \frac{1}{\lambda} \mathcal{Z}_\Delta, g \rangle$, $b_2 := \langle \mathcal{Z}, \Lambda W \rangle^{-1} \cdot \langle \mathcal{Z}, g \rangle$ and decompose

$$g = a_1^- \mathcal{Y}_\lambda^- + a_1^+ \mathcal{Y}_\lambda^+ + a_2^- \mathcal{Y}^- + a_2^+ \mathcal{Y}^+ + b_1 \Lambda W_\lambda + b_2 \Lambda W + k.$$

Using the fact that

$$\begin{aligned} |\langle \alpha^\pm, \mathcal{Y}_\lambda^\pm \rangle| + |\langle \alpha_\lambda^\pm, \mathcal{Y}^\pm \rangle| + |\langle \frac{1}{\lambda} \mathcal{Z}_\Delta, \mathcal{Y} \rangle| + |\langle \mathcal{Z}, \mathcal{Y}_\lambda \rangle| &\lesssim \lambda^2, \\ |a_1^-| + |a_1^+| + |a_2^-| + |a_2^+| + |b_1| + |b_2| &\lesssim \|g\|_{\mathcal{E}}, \\ \langle \alpha^-, \mathcal{Y}^+ \rangle = \langle \alpha^+, \mathcal{Y}^- \rangle = \langle \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Y}, \Lambda W \rangle &= 0 \end{aligned}$$

we obtain

$$(3.33) \quad \langle \alpha^-, k \rangle^2 + \langle \alpha^+, k \rangle^2 + \langle \alpha_\lambda^-, k \rangle^2 + \langle \alpha_\lambda^+, k \rangle^2 + \langle \mathcal{Z}, k \rangle^2 + \langle \frac{1}{\lambda} \mathcal{Z}_\Delta, k \rangle^2 \lesssim \lambda^4 \cdot \|g\|_{\mathcal{E}}^2.$$

Since L_λ is self-adjoint, we can write

$$\begin{aligned} (3.34) \quad \frac{1}{2} \langle L_\lambda g, g \rangle &= \frac{1}{2} \langle L_\lambda k, k \rangle \\ &+ \langle L_\lambda (a_2^- \mathcal{Y}^- + a_2^+ \mathcal{Y}^+ + b_2 \Lambda W), k \rangle + \langle L_\lambda (a_1^- \mathcal{Y}_\lambda^- + a_1^+ \mathcal{Y}_\lambda^+ + b_1 \Lambda W_\lambda), k \rangle \\ &+ \frac{1}{2} \langle L_\lambda (a_2^- \mathcal{Y}^- + a_2^+ \mathcal{Y}^+ + b_2 \Lambda W), a_2^- \mathcal{Y}^- + a_2^+ \mathcal{Y}^+ + b_2 \Lambda W \rangle \\ &+ \frac{1}{2} \langle L_\lambda (a_1^- \mathcal{Y}_\lambda^- + a_1^+ \mathcal{Y}_\lambda^+ + b_1 \Lambda W_\lambda), a_1^- \mathcal{Y}_\lambda^- + a_1^+ \mathcal{Y}_\lambda^+ + b_1 \Lambda W_\lambda \rangle \\ &+ \langle L_\lambda (a_2^- \mathcal{Y}^- + a_2^+ \mathcal{Y}^+ + b_2 \Lambda W), a_1^- \mathcal{Y}_\lambda^- + a_1^+ \mathcal{Y}_\lambda^+ + b_1 \Lambda W_\lambda \rangle. \end{aligned}$$

It is easy to see that $\|f'(W)\mathcal{Y}_\lambda\|_{L^{\frac{3}{2}}} \rightarrow 0$, $\|f'(W)\Lambda W_\lambda\|_{L^{\frac{3}{2}}} \rightarrow 0$, $\|f'(W_\lambda)\mathcal{Y}\|_{L^{\frac{3}{2}}} \rightarrow 0$ and $\|f'(W_\lambda)\Lambda W\|_{L^{\frac{3}{2}}} \rightarrow 0$ as $\lambda \rightarrow 0$. This and (3.2), (3.3) imply

$$\begin{aligned} & \|L_\lambda \mathcal{Y}^- + 2\alpha^+ \|_{\mathcal{E}^*} + \|L_\lambda \mathcal{Y}^+ + 2\alpha^- \|_{\mathcal{E}^*} + \|L_\lambda \Lambda W\|_{\mathcal{E}^*} \\ & + \|L_\lambda \mathcal{Y}_\lambda^- + 2\alpha_\lambda^+ \|_{\mathcal{E}^*} + \|L_\lambda \mathcal{Y}_\lambda^+ + 2\alpha_\lambda^- \|_{\mathcal{E}^*} + \|L_\lambda \Lambda W_\lambda\|_{\mathcal{E}^*} \xrightarrow{\lambda \rightarrow 0} 0. \end{aligned}$$

Plugging this into (3.34) and using (3.33) we obtain

$$(3.35) \quad \frac{1}{2} \langle L_\lambda g, g \rangle \geq -2a_2^- a_2^+ - 2a_1^- a_1^+ + \frac{1}{2} \langle L_\lambda k, k \rangle - \tilde{c} \|g\|_{\mathcal{E}}^2,$$

where $\tilde{c} \rightarrow 0$ as $\lambda \rightarrow 0$.

Applying (3.30) with $r_1 = \lambda^{-\frac{1}{2}}$, rescaling and using (3.33) we get, for λ small enough,

$$(3.36) \quad (1 - 2c) \int_{|x| \leq \sqrt{\lambda}} |\nabla k|^2 dx + c \int_{|x| \geq \sqrt{\lambda}} |\nabla k|^2 dx - \int_{\mathbb{R}^6} f'(W_\lambda) |k|^2 dx \geq -\tilde{c} \|g\|_{\mathcal{E}}^2.$$

From (3.31) with $r_2 = \sqrt{\lambda}$ we have

$$(3.37) \quad (1 - 2c) \int_{|x| \geq \sqrt{\lambda}} |\nabla k|^2 dx + c \int_{|x| \leq \sqrt{\lambda}} |\nabla k|^2 dx - \int_{\mathbb{R}^6} f'(W) |k|^2 dx \geq -\tilde{c} \|g\|_{\mathcal{E}}^2.$$

Taking the sum of (3.36) and (3.37), and using (3.35), we obtain

$$\frac{1}{2} \langle L_\lambda g, g \rangle \geq -2a_2^- a_2^+ - 2a_1^- a_1^+ + c \|k\|_{\mathcal{E}}^2 - 2\tilde{c} \|g\|_{\mathcal{E}}^2.$$

The conclusion follows if we take \tilde{c} small enough. \square

3.4. Definition of the mixed energy-virial functional.

Lemma 3.10. *For any $c > 0$ and $R > 0$ there exists a radial function $q(x) = q_{c,R}(x) \in C^{3,1}(\mathbb{R}^6)$ with the following properties:*

- (P1) $q(x) = \frac{1}{2}|x|^2$ for $|x| \leq R$,
- (P2) there exists $\tilde{R} > 0$ (depending on c and R) such that $q(x) \equiv \text{const}$ for $|x| \geq \tilde{R}$,
- (P3) $|\nabla q(x)| \lesssim |x|$ and $|\Delta q(x)| \lesssim 1$ for all $x \in \mathbb{R}^6$, with constants independent of c and R ,
- (P4) $\sum_{1 \leq i, j \leq 6} (\partial_{x_i x_j} q(x)) v_i v_j \geq -c \sum_{i=1}^6 v_i^2$, for all $x \in \mathbb{R}^6$, $v_i \in \mathbb{R}$,
- (P5) $\Delta^2 q(x) \leq c \cdot |x|^{-2}$, for all $x \in \mathbb{R}^6$.

Remark 3.11. We require $C^{3,1}$ regularity in order not to worry about boundary terms in Pohozaev identities, see the proof of (3.41).

Proof. It suffices to prove the result for $R = 1$ since the function $q_R(x) := R^2 q(\frac{x}{R})$ satisfies the listed properties if and only if $q(x)$ does.

Let r denote the radial coordinate. Define $q_0(x)$ by the formula

$$q_0(r) := \begin{cases} \frac{1}{2} \cdot r^2 & r \leq 1 \\ \frac{8}{5} \cdot r - \frac{3}{2} + \frac{1}{2} \cdot r^{-2} - \frac{1}{10} \cdot r^{-4} & r \geq 1. \end{cases}$$

A direct computation shows that for $r > 1$ we have $q_0'(r) = \frac{8}{5} - r^{-3} + \frac{2}{5}r^{-5}$, $q_0''(r) = 3r^{-4} - 2r^{-6} > 0$ (so $q_0(x)$ is convex), $q_0'''(r) = 12(-r^{-5} + r^{-7})$ and $\Delta^2 q_0(r) = -24r^{-3}$. Hence q_0 satisfies all the listed properties except for (P2). We correct it as follows.

Let $e_j(r) := \frac{1}{j!} r^j \cdot \chi(r)$ for $j \in \{1, 2, 3\}$ and let $R_0 \gg 1$. We define

$$q(r) := \begin{cases} q_0(r) & r \leq R_0 \\ q_0(R_0) + \sum_{j=1}^3 q_0^{(j)}(R_0) \cdot R_0^j \cdot e_j(-1 + R_0^{-1}r) & r \geq R_0. \end{cases}$$

Note that $q'_0(R_0) \sim 1$, $q''_0(R_0) \sim R_0^{-4}$ and $q'''_0(R_0) \sim R_0^{-5}$. It is clear that $q(x) \in C^{3,1}(\mathbb{R}^6)$. Property (P1) holds since $R_0 > 1$. By the definition of the functions e_j we have $q(r) = q_0(R_0) = \text{const}$ for $r \geq 3R_0$, hence (P2) holds with $\tilde{R} = 3R_0$. From the definition of $q(r)$ we get $|\partial_r q(r)| \lesssim 1$ and $|\partial_r^2 q(r)| \lesssim R_0^{-4}$ for $r \geq R_0$, with a constant independent of R_0 , which implies (P3). Similarly, $|\partial_{x_i x_j} q(x)| \lesssim R_0^{-1}$ for $|x| \geq R_0$, which implies (P4) if R_0 is large enough. Finally $|\Delta^2 q(x)| \lesssim R_0^{-3}$ for $|x| \geq R_0$ and $\Delta^2 q(x) = 0$ for $|x| \geq 3R_0$. This proves (P5) if R_0 is large enough. \square

In the sequel $q(x)$ always denotes a function of class $C^{3,1}(\mathbb{R}^6)$ verifying (P1)–(P5) with sufficiently small c and sufficiently large R .

For $\lambda > 0$ we define the operators $A(\lambda)$ and $A_0(\lambda)$ as follows:

$$(3.38) \quad \begin{aligned} [A(\lambda)h](x) &:= \frac{1}{3\lambda} \Delta q\left(\frac{x}{\lambda}\right) h(x) + \nabla q\left(\frac{x}{\lambda}\right) \cdot \nabla h(x), \\ [A_0(\lambda)h](x) &:= \frac{1}{2\lambda} \Delta q\left(\frac{x}{\lambda}\right) h(x) + \nabla q\left(\frac{x}{\lambda}\right) \cdot \nabla h(x). \end{aligned}$$

Combining these definitions with the fact that $q(x)$ is an approximation of $\frac{1}{2}|x|^2$ we see that $A(\lambda)$ and $A_0(\lambda)$ are approximations (in a sense not yet precised) of $\frac{1}{\lambda}\Lambda$ and $\frac{1}{\lambda}\Lambda_0$ respectively. We will write A and A_0 instead of $A(1)$ and $A_0(1)$ respectively. Note the following scale-change formulas, which follow directly from the definitions:

$$(3.39) \quad \forall h \in \dot{H}^1 : \quad A(\lambda)(h_\lambda) = (Ah)_\Delta, \quad A_0(\lambda)(h_\lambda) = (A_0 h)_\Delta.$$

Lemma 3.12. *The operators $A(\lambda)$ and $A_0(\lambda)$ have the following properties:*

- the families $\{A(\lambda) : \lambda > 0\}$, $\{A_0(\lambda) : \lambda > 0\}$, $\{\lambda \partial_\lambda A(\lambda) : \lambda > 0\}$ and $\{\lambda \partial_\lambda A_0(\lambda) : \lambda > 0\}$ are bounded in $\mathcal{L}(\dot{H}^1; L^2)$, with the bound depending on the choice of the function $q(x)$,
- for all $h_1, h_2 \in X^1$ and $\lambda > 0$ there holds

$$(3.40) \quad \langle A(\lambda)h_1, f(h_1 + h_2) - f(h_1) - f'(h_1)h_2 \rangle = -\langle A(\lambda)h_2, f(h_1 + h_2) - f(h_1) \rangle,$$

- for any $c_0 > 0$, if we choose c in Lemma 3.10 small enough, then for all $h \in X^1$ there holds

$$(3.41) \quad \langle A_0(\lambda)h, \Delta h \rangle \leq \frac{c_0}{\lambda} \|h\|_{\dot{H}^1}^2 - \frac{1}{\lambda} \int_{|x| \leq R\lambda} |\nabla h(x)|^2 dx.$$

- assuming (2.7) and (3.10), for any $c_0 > 0$ there holds

$$(3.42) \quad \|\Lambda_0 \Lambda W_{\lambda(t)} - A_0(\lambda(t)) \Lambda W_{\lambda(t)}\|_{L^2} \leq c_0,$$

$$(3.43) \quad \|\dot{\varphi}(t) + b(t) \cdot A(\lambda(t))\varphi(t)\|_{L^3} \leq c_0,$$

$$(3.44) \quad \left| \int \frac{1}{6} \Delta q\left(\frac{x}{\lambda}\right) (f(\varphi + g) - f(\varphi))g dx - \int f'(W_\lambda)g^2 dx \right| \leq c_0 C_0^2 e^{-3\kappa|t|}.$$

provided that the constant R in the definition of $q(x)$ is chosen large enough.

Proof. Since $\nabla q(x)$ and $\nabla^2 q(x)$ are continuous and of compact support, it is clear that A and A_0 are bounded operators $\dot{H}^1 \rightarrow L^2$. From the invariance (3.39) we see that $A(\lambda)$ and $A_0(\lambda)$ have the same norms as A and A_0 respectively. For $\lambda \partial_\lambda A(\lambda)$ and $\lambda \partial_\lambda A_0(\lambda)$ the proof is similar. We compute

$$\partial_\lambda A(\lambda) = -\frac{1}{3\lambda^2} \Delta q\left(\frac{x}{\lambda}\right) - \frac{1}{3\lambda^3} x \cdot \nabla \Delta q\left(\frac{x}{\lambda}\right) - \frac{1}{\lambda^2} x \cdot \nabla^2 q\left(\frac{x}{\lambda}\right) \cdot \nabla.$$

Since $\nabla q(x)$, $\nabla^2 q(x)$ and $\nabla^3 q(x)$ are continuous and of compact support, boundedness follows.

In (3.40) both sides are continuous for the X^1 topology, hence we may assume that $h_1, h_2 \in C_0^\infty$. We may also assume without loss of generality that $\lambda = 1$. Observe that for any $h \in C_0^\infty$ there

holds $h \cdot f(h) = 3 \cdot F(h)$ and $\nabla h \cdot f(h) = \nabla F(h)$, hence

$$\langle Ah, f(h) \rangle = \int \left(\frac{1}{3} \Delta q \cdot h + \nabla q \cdot \nabla h \right) f(h) \, dx = \int \Delta q \cdot F(h) + \nabla q \cdot \nabla F(h) \, dx = 0.$$

Using this for $h = h_1 + h_2$ and for $h = h_1$, (3.40) is seen to be equivalent to

$$(3.45) \quad \langle Ah_2, f(h_1) \rangle + \langle Ah_1, f'(h_1)h_2 \rangle = 0.$$

Expanding the left side using the definition of A we obtain

$$\begin{aligned} \langle Ah_2, f(h_1) \rangle + \langle Ah_1, f'(h_1)h_2 \rangle &= \int \frac{1}{3} \Delta q \cdot h_2 \cdot f(h_1) + \nabla q \cdot \nabla h_2 \cdot f(h_1) \, dx \\ &\quad + \int \frac{1}{3} \Delta q \cdot h_1 \cdot f'(h_1) \cdot h_2 + \nabla q \cdot \nabla h_1 \cdot f'(h_1) \cdot h_2 \, dx \end{aligned}$$

Integrating by parts the term containing ∇h_2 and using the formulas $h_1 \cdot f'(h_1) = 2f(h_1)$ and $\nabla h_1 \cdot f'(h_1) = \nabla f(h_1)$, this can be rewritten as

$$\left\langle h_2, \frac{1}{3} \Delta q \cdot f(h_1) - \Delta q \cdot f(h_1) - \nabla q \cdot \nabla f(h_1) + \frac{2}{3} \Delta q \cdot f(h_1) + \nabla q \cdot \nabla f(h_1) \right\rangle = 0,$$

which proves (3.45).

Inequality (3.41) follows easily from (P1), (P4) and (P5), once we check the following identity (valid in any dimension N , and used here for $N = 6$):

$$(3.46) \quad \begin{aligned} &\int \Delta h(x) \cdot \left(\frac{1}{2\lambda} \Delta q \left(\frac{x}{\lambda} \right) h(x) + \nabla q \left(\frac{x}{\lambda} \right) \cdot \nabla h(x) \right) \, dx \\ &= -\frac{1}{4\lambda} \int (\Delta^2 q) \left(\frac{x}{\lambda} \right) h(x)^2 \, dx - \frac{1}{\lambda} \int \sum_{i,j=1}^N \partial_{ij} q \left(\frac{x}{\lambda} \right) \partial_i h(x) \partial_j h(x) \, dx. \end{aligned}$$

Without loss of generality we can assume that $\lambda = 1$ (it suffices to replace q by its rescaled version). By a density argument, we can also assume that $q, h \in C_0^\infty$ (we use here the fact that $q \in C^{3,1}$), and (3.46) follows from integration by parts:

$$\begin{aligned} &\int \frac{1}{2} \Delta h \cdot \Delta q \cdot h + \Delta h \cdot \nabla q \cdot \nabla h \, dx = \int \sum_{i,j=1}^N \left(\frac{1}{2} \partial_{ii} h \cdot \partial_{jj} q \cdot h + \partial_{ii} h \cdot \partial_j q \cdot \partial_j h \right) \, dx \\ &= \int -\frac{1}{2} \sum_{i,j} \partial_i h (\partial_{jj} q \partial_i h + \partial_{ij} q \cdot h) + \sum_i \frac{1}{2} \partial_i ((\partial_i h)^2) \partial_i q \\ &\quad + \sum_{i \neq j} \left(-\frac{1}{2} \partial_j (\partial_i h)^2 \partial_j q - \partial_{ij} q \partial_i h \partial_j h \right) \, dx \\ &= \int -\frac{1}{2} \sum_{i,j} (\partial_{jj} q (\partial_i h)^2 + \frac{1}{2} \partial_{iijj} q \cdot h^2) - \frac{1}{2} \sum_i \partial_{ii} q (\partial_i h)^2 \\ &\quad + \frac{1}{2} \sum_{i \neq j} \partial_{jj} q (\partial_i h)^2 - \sum_{i \neq j} \partial_{ij} q \partial_i h \partial_j h \, dx \\ &= \int -\frac{1}{4} \sum_{i,j} \partial_{iijj} q \cdot h^2 - \sum_{i,j} \partial_{ij} q \partial_i h \partial_j h \, dx. \end{aligned}$$

Estimate (3.42) is invariant by rescaling, hence we can assume that $\lambda = 1$. For $|x| \leq R$ we have $A_0 \Lambda W(x) = \Lambda_0 \Lambda W(x)$. From (P3) in Lemma 3.10 we get $|A_0 \Lambda W(x)| + |\Lambda_0 \Lambda W(x)| \lesssim |x|^{-4}$ for $|x| \geq R$, with a constant independent of R . Thus $\|\Lambda_0 \Lambda W - A_0 \Lambda W\|_{L^2} \leq c_0$ if R is large enough.

A similar reasoning yields $\|\Lambda W_{\underline{\lambda}} - A(\lambda)W_{\lambda}\|_{L^3} \ll \lambda^{-1}$ as $R \rightarrow +\infty$. Since $b(t) \sim \lambda(t)$, this gives

$$(3.47) \quad \|\dot{\varphi} + bA(\lambda)W_{\lambda}\|_{L^3} \leq \frac{c_0}{3}, \quad \text{if } R \text{ is large enough.}$$

From (P2) in Lemma 3.10 it follows that $\text{supp}(A(\lambda)W_{\mu}) \subset B(0, \tilde{R} \cdot \lambda)$. Since $\|A(\lambda)W_{\mu}\|_{L^\infty} \lesssim_q \frac{1}{\lambda}$, we have

$$(3.48) \quad \|bA(\lambda)W_{\mu}\|_{L^3} \leq \frac{c_0}{3}, \quad \text{if } |T_0| \text{ is large enough.}$$

To finish the proof, we have to check that

$$(3.49) \quad \|bA(\lambda)(\chi \cdot (\lambda(t)^2 P_{\lambda(t)} + b(t)^2 Q_{\lambda(t)}))\|_{L^3} \leq \frac{c_0}{3}, \quad \text{if } |T_0| \text{ is large enough.}$$

We have the bound $\|(\chi + |\nabla\chi|)P_{\lambda}\|_{L^3} \lesssim \int_0^{\frac{2}{\lambda}} \frac{1}{r^6} r^5 dr \lesssim |\log \lambda|$ (similarly with Q_{λ}), hence $\|(\chi + |\nabla\chi|)(\lambda^2 P_{\lambda} + b^2 Q_{\lambda})\|_{L^3} \lesssim (\lambda^2 + b^2)|\log \lambda| \ll 1$ as $|T_0| \rightarrow +\infty$. We have also $\|\nabla(\lambda^2 P_{\lambda} + b^2 Q_{\lambda})\|_{L^3} \lesssim \frac{1}{\lambda}(\lambda^2 + b^2) \ll 1$. Since q is smooth and constant at infinity, we have $|b \cdot (\frac{1}{\lambda} \Delta q(\frac{x}{\lambda}) + \nabla q(\frac{x}{\lambda}))| \lesssim 1$. The constant depends on the choice of the function q , but this is not a concern here. We obtain

$$\begin{aligned} & \|bA(\lambda)(\chi \cdot (\lambda(t)^2 P_{\lambda(t)} + b(t)^2 Q_{\lambda(t)}))\|_{L^3} \lesssim |b \cdot (\frac{1}{\lambda} \Delta q(\frac{x}{\lambda}) + \nabla q(\frac{x}{\lambda}))| \\ & \cdot (\|(\chi + |\nabla\chi|)(\lambda^2 P_{\lambda} + b^2 Q_{\lambda})\|_{L^3} + \|\nabla(\lambda^2 P_{\lambda} + b^2 Q_{\lambda})\|_{L^3}) \ll 1, \end{aligned}$$

hence (3.49)

Putting together (3.47), (3.48) and (3.49) we get (3.43).

In order to prove (3.44), note first that boundedness of Δq and (3.23) yield

$$\left| \int \frac{1}{6} \Delta q\left(\frac{x}{\lambda}\right) (f(\varphi + g) - f(\varphi)) g \, dx - \int \frac{1}{6} \Delta q\left(\frac{x}{\lambda}\right) (f'(W_{\mu}) + f'(W_{\lambda})) g^2 \, dx \right| \ll e^{-3\kappa|t|}.$$

Since ∇q is of compact support, we have $\left| \int \frac{1}{6} \Delta q\left(\frac{x}{\lambda}\right) W_{\mu} g^2 \, dx \right| \ll \|g\|_{\dot{H}^1}^2$. As in the proof of (3.43), one can show that $\left\| \frac{1}{6} \Delta q\left(\frac{x}{\lambda}\right) f'(W_{\lambda}) - f'(W_{\lambda}) \right\|_{L^3} \rightarrow 0$ as $R \rightarrow +\infty$. This finishes the proof. \square

For $t \in [T, T_0]$ we define:

- the *nonlinear energy functional*

$$\begin{aligned} I(t) &:= \int \frac{1}{2} |\dot{g}(t)|^2 + \frac{1}{2} |\nabla g(t)|^2 - (F(\varphi(t) + g(t)) - F(\varphi(t)) - f(\varphi(t))g(t)) \, dx \\ &= E(\varphi(t) + \mathbf{g}(t)) - E(\varphi(t)) - \langle DE(\varphi(t)), \mathbf{g}(t) \rangle, \end{aligned}$$

- the *localized virial functional*

$$J(t) := \int \dot{g}(t) \cdot A_0(\lambda(t)) g(t) \, dx,$$

- the *mixed energy-virial functional*

$$H(t) := I(t) + b(t)J(t).$$

From (2.16) we have

$$|I(t) - \frac{1}{2} \langle D^2 E(\varphi(t)) \mathbf{g}(t), \mathbf{g}(t) \rangle| \lesssim \|\mathbf{g}(t)\|_{\mathcal{E}}^3.$$

Note that $H(t)$ depends on the function $q(x)$ used in the definition of $A_0(\lambda)$, see (3.38). From the first statement in Lemma 3.12 we deduce that

$$|J(t)| \lesssim_q \|\mathbf{g}(t)\|_{\mathcal{E}}^2,$$

where the constant in the inequality depends on the choice of the function $q(x)$. Thus (2.9) and (3.10) imply that for $t \leq T_0$ with $|T_0|$ large enough there holds

$$(3.50) \quad \left| H(t) - \frac{1}{2} \langle D^2 E(\varphi(t)) \mathbf{g}(t), \mathbf{g}(t) \rangle \right| \leq c \|\mathbf{g}(t)\|_{\mathcal{E}}^2,$$

with $c > 0$ as small as we wish.

3.5. Energy estimates via the mixed energy-virial functional.

Lemma 3.13. *Let $c_1 > 0$. If C_0 is sufficiently large, then there exists a function $q(x)$ and $T_0 < 0$ with the following property. If $T_1 < T_0$ and (2.7), (3.10), (3.11) hold for $t \in [T, T_1]$, then for $t \in [T, T_1]$ there holds*

$$(3.51) \quad H'(t) \leq c_1 \cdot C_0^2 \cdot e^{-3\kappa|t|}.$$

This lemma is the key step in proving Proposition 3.3. We will postpone its slightly technical proof.

Proof of Proposition 3.3 assuming Lemma 3.13. We first show (3.13). From (3.15) and (3.10) we obtain

$$(3.52) \quad |\mu(t) - 1| = |\mu(t) - \mu(T)| \lesssim \int_{-\infty}^t C_0 \cdot e^{-\frac{3}{2}\kappa_0|\tau|} d\tau \lesssim C_0 \cdot e^{-\frac{3}{2}\kappa_0|t|}.$$

Again from (3.15) and (3.10) we have $|\lambda'(t) - b(t)| \lesssim C_0 \cdot e^{-\frac{3}{2}\kappa|t|}$. Multiplying by $b'(t) = \kappa^2 \cdot \frac{\lambda(t)}{\mu(t)^2} \sim e^{-\kappa|t|}$, cf. (2.8) and (2.7), we obtain $\left| \frac{d}{dt} (b(t)^2 - \kappa^2 \cdot \frac{\lambda(t)^2}{\mu(t)^2}) \right| \lesssim C_0 \cdot e^{-\frac{5}{2}\kappa|t|}$. Since $b(T) = \kappa \cdot \lambda(T)$ and $\mu(T) = 1$, this yields $|b(t)^2 - \kappa^2 \cdot \frac{\lambda(t)^2}{\mu(t)^2}| \lesssim C_0 \cdot e^{-\frac{5}{2}\kappa|t|}$. But $b(t) + \kappa \cdot \frac{\lambda(t)}{\mu(t)} \sim e^{-\kappa|t|}$, see (2.7) and (2.9), hence

$$(3.53) \quad \left| b(t) - \kappa \cdot \frac{\lambda(t)}{\mu(t)} \right| \lesssim C_0 \cdot e^{-\frac{3}{2}\kappa|t|}.$$

Bound (3.52) implies that $\left| \frac{\lambda(t)}{\mu(t)} - \lambda(t) \right| \ll e^{-\frac{3}{2}\kappa|t|}$, thus (3.53) yields $|\lambda'(t) - \kappa \cdot \lambda(t)| \lesssim C_0 \cdot e^{-\frac{3}{2}\kappa|t|}$. Integrating and using $\lambda(T) = \frac{1}{\kappa} e^{-\kappa|T|}$ we obtain (3.13).

We turn to the proof of (3.12). The initial data at $t = T$ satisfy $\|\mathbf{g}(T)\|_{\mathcal{E}} \lesssim e^{-\frac{3}{2}\kappa|T|}$, thus (3.50) implies that $H(T) \lesssim e^{-3\kappa|T|}$. If C_0 is large enough, then integrating (3.51) we get $H(t) \leq c \cdot C_0^2 \cdot e^{-3\kappa|t|}$, with a small constant c . Now (3.50) implies

$$(3.54) \quad \langle D^2 E(\varphi(t)) \mathbf{g}(t), \mathbf{g}(t) \rangle \lesssim c \cdot C_0^2 \cdot e^{-3\kappa|t|}, \quad \text{with } c \text{ small.}$$

Since $\|\varphi(t) - \mathbf{W}_{\lambda(t)}\|_{\mathcal{E}}$ is small, Lemma 3.9 together with (3.14), (3.11), (3.18) and (3.19) yields

$$\|\mathbf{g}\|_{\mathcal{E}}^2 \lesssim (cC_0^2 + 1)e^{-3\kappa|t|},$$

Eventually enlarging C_0 , we obtain (3.12), if c in (3.54) is taken sufficiently small. \square

Proof of Lemma 3.13. In this proof we say that a term is negligible if its contribution is $\leq c \cdot C_0^2 \cdot e^{-3\kappa|t|}$. We write $\text{Value}_1 \simeq \text{Value}_2$ if $|\text{Value}_1 - \text{Value}_2| \leq c \cdot C_0^2 \cdot e^{-3\kappa|t|}$. The order of choosing the parameters is the following: we will first choose $q(x)$ independently of C_0 , then C_0 , which may depend on $q(x)$, and finally $|T_0|$.

Step 1 (Derivative of the energy functional). Using the definition of $I(t)$, the conservation of energy, formulas (2.13), (3.9) and self-adjointness of $D^2E(\varphi)$ we compute:

$$\begin{aligned}
I'(t) &= 0 - \langle DE(\varphi), \partial_t \varphi \rangle - \langle D^2E(\varphi) \partial_t \varphi, g \rangle - \langle DE(\varphi), \partial_t g \rangle \\
&= -\langle DE(\varphi), J \circ DE(\varphi) + \psi \rangle - \langle J \circ DE(\varphi) + \psi, D^2E(\varphi) g \rangle \\
&\quad - \langle DE(\varphi), J \circ (DE(\varphi + g) - DE(\varphi)) - \psi \rangle \\
&= -\langle D^2E(\varphi) \psi, g \rangle - \langle DE(\varphi), J \circ (DE(\varphi + g) - DE(\varphi) - D^2E(\varphi) g) \rangle \\
&= \langle (\Delta + f'(\varphi)) \psi, g \rangle - \langle \dot{\psi}, \dot{g} \rangle - \langle \dot{\varphi}, f(\varphi + g) - f(\varphi) - f'(\varphi) g \rangle.
\end{aligned}$$

The first term is $\lesssim C_0 e^{-3\kappa|t|}$, due to (2.19) and (3.10), hence it is negligible (by enlarging C_0 if necessary). Inequality (2.18) implies that the second term can be replaced by $-\frac{b}{\lambda}(\lambda' - b) \langle \Lambda_0 \Lambda W_{\underline{\lambda}}, \dot{g} \rangle$, which in turn can be replaced by $-b(\lambda' - b) \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle$, thanks to (3.42). From (3.43) we infer that the third term can be replaced by $b \cdot \langle A(\lambda) g, f(\varphi + g) - f(\varphi) - f'(\varphi) g \rangle$. Using formula (3.40) with $h_1 = \varphi$ and $h_2 = g$ we obtain

$$(3.55) \quad I'(t) \simeq -b(\lambda' - b) \cdot \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle - b \cdot \langle A(\lambda) g, f(\varphi + g) - f(\varphi) \rangle.$$

Step 2 (Derivative of the virial functional). We compute:

$$\begin{aligned}
(bJ)'(t) &= b' \int \dot{g} \cdot A_0(\lambda) g \, dx + b \lambda' \int \dot{g} \cdot \partial_\lambda A_0(\lambda) g \, dx \\
(3.56) \quad &\quad + b \int \dot{g} \cdot A_0(\lambda) (\dot{g} - \psi) \, dx + b \int (\Delta g + f(\varphi + g) - f(\varphi) - \dot{\psi}) \cdot A_0(\lambda) g \, dx.
\end{aligned}$$

The first two terms are negligible thanks to Lemma 3.12. Consider the third term on the right in (3.56). An integration by parts yields $\int \dot{g} \cdot A_0(\lambda) \dot{g} \, dx = 0$. Since $A_0(\lambda) : \dot{H}^1 \rightarrow L^2$ is a bounded operator, from (2.17) we see that

$$b \int \dot{g} \cdot A_0(\lambda) \psi \, dx \simeq -b(\lambda' - b) \cdot \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle.$$

Consider the forth term on the right in (3.56). The term $b \int \dot{\psi} \cdot A_0(\lambda) g \, dx$ is negligible. Using (3.41) and the fact that $A_0(\lambda) = A(\lambda) + \frac{1}{6\lambda} \Delta q(\frac{\cdot}{\lambda})$ we get

$$\begin{aligned}
b \int (\Delta g + f(\varphi + g) - f(\varphi)) \cdot A_0(\lambda) g \, dx &\leq b \cdot \langle A(\lambda) g, f(\varphi + g) - f(\varphi) \rangle \\
&\quad - \frac{b}{\lambda} \int_{|x| \leq R\lambda} |\nabla g|^2 \, dx + \frac{b}{\lambda} \int \frac{1}{6} \Delta q(\frac{\cdot}{\lambda}) (f(\varphi + g) - f(\varphi)) g \, dx + cC_0^2 e^{-3\kappa|t|},
\end{aligned}$$

with a small constant c . Putting everything back together and using (3.44) we get

$$\begin{aligned}
(bJ)'(t) &\leq b(\lambda' - b) \cdot \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle + b \cdot \langle A(\lambda) g, f(\varphi + g) - f(\varphi) \rangle \\
&\quad + \frac{b}{\lambda} \left(- \int_{|x| \leq R\lambda} |\nabla g|^2 \, dx + \int f'(W_\lambda) g^2 \, dx \right) + cC_0^2 e^{-3\kappa|t|}.
\end{aligned}$$

Step 3 (Localized coercivity). Taking the sum of (3.55) and (3.56) we obtain

$$H'(t) \leq \frac{b}{\lambda} \left(- \int_{|x| \leq R\lambda} |\nabla g|^2 \, dx + \int f'(W_\lambda) g^2 \, dx \right) + cC_0^2 e^{-3\kappa|t|}.$$

Recall that $|\langle \frac{1}{\lambda} \mathcal{Y}_{\underline{\lambda}}, g \rangle| \lesssim |a_1^+| + |a_1^-|$, hence (3.30) (after rescaling) together with (3.14), (3.11) and (3.18) imply that

$$- \int_{|x| \leq R\lambda} |\nabla g|^2 \, dx + \int f'(W_\lambda) g^2 \, dx \lesssim (cC_0^2 + 1) e^{-3\kappa|t|},$$

with $c > 0$ as small as we wish (by taking R large enough). Enlarging C_0 if necessary we arrive at (3.51). \square

3.6. Shooting argument and passing to a limit. We are ready to give a proof of the main result of the paper, following a well-known scheme introduced in [32] and [29].

Proof of Theorem 1.

Step 1. Let T_n be a decreasing sequence converging to $-\infty$. For n large and

$$a_0 \in \mathcal{A} := \left[-\frac{1}{2}e^{-\frac{3}{2}\kappa|T_n|}, \frac{1}{2}e^{-\frac{3}{2}\kappa|T_n|} \right],$$

let $\mathbf{u}_n^{a_0}(t) : [T_n, T_+] \rightarrow \mathcal{E}$ denote the solution of (1.1) with initial data (3.5). We will prove that there exists a_0 such that $T_+ > T_0$ and for $\mathbf{u} = \mathbf{u}_n^{a_0}$ inequalities (3.12), (3.13) hold for $t \in [T_n, T_0]$.

Suppose that this is not the case. For each $a_0 \in \mathcal{A}$, let $T_1 = T_1(a_0)$ be the last time such that (3.12) and (3.13) hold for $t \in [T_n, T_1]$. Since $\{\varphi(t) : t \in [T_n, T_1]\}$ is a compact set, Corollary A.4 implies that $T_+ > T_1$. Suppose that $|a_1^+(T_1)| < e^{-\frac{3}{2}\kappa|T_1|}$. Then Proposition 3.3 would imply that (3.12) and (3.13) hold on some neighborhood of T_1 , contradicting its definition. Thus $a_1^+(T_1) = e^{-\frac{3}{2}\kappa|T_1|}$ or $a_1^+(T_1) = -e^{-\frac{3}{2}\kappa|T_1|}$. Let $\mathcal{A}_+ \subset \mathcal{A}$ be the set of $a_0 \in \mathcal{A}$ which lead to $a_1^+(T_1) = e^{-\frac{3}{2}\kappa|T_1|}$ and let $\mathcal{A}_- \subset \mathcal{A}$ be the set of $a_0 \in \mathcal{A}$ which lead to $a_1^+(T_1) = -e^{-\frac{3}{2}\kappa|T_1|}$. We have proved that $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$. We will show that \mathcal{A}_+ and \mathcal{A}_- are open sets, which will lead to a contradiction since \mathcal{A} is connected.

Let $a_0 \in \mathcal{A}_+$. This implies that there exists the first T_2 such that $a_1^+(T_2) > \frac{3}{4}e^{-\frac{3}{2}\kappa|T_2|}$. Hence for a solution $\tilde{\mathbf{u}}(t)$ corresponding to \tilde{a}_0 close to a_0 we will have (by continuity of the flow) $\tilde{a}_1^+(T_2) > \frac{3}{4}e^{-\frac{3}{2}\kappa|T_2|}$ and $|\tilde{a}_1^+(t)| < e^{-\frac{3}{2}\kappa|t|}$ for $t \in [T_n, T_2]$. Suppose that $\tilde{a}_0 \in \mathcal{A}_-$. Hence there exists the first $T_3 > T_2$ such that $\tilde{a}_1^+(T_3) = \frac{3}{4}e^{-\frac{3}{2}\kappa|T_3|}$. Estimate (3.17) yields $\tilde{a}_1^+(T_3) \gtrsim e^{-\frac{1}{2}\kappa|T_3|} \gg e^{-\frac{3}{2}\kappa|T_3|}$, which is a contradiction. Hence \mathcal{A}_+ is open and analogously \mathcal{A}_- is open.

Step 2. Call \mathbf{u}_n the solution found in Step 1. From (3.12), (3.13) and (2.11) we deduce that there exists a constant $C_1 > 0$ independent of n such that

$$(3.57) \quad \|\mathbf{u}_n(t) - (W_{\frac{1}{\kappa}e^{-\kappa|t|}} - W, -e^{-\kappa|t|}\Delta W_{\frac{1}{\kappa}e^{-\kappa|t|}})\|_{\mathcal{E}} \leq C_1 \cdot e^{-\frac{3}{2}\kappa|t|}, \quad \text{for } t \in [T_n, T_0].$$

The sequence $\mathbf{u}_n(T_0)$ is bounded in \mathcal{E} , hence its subsequence (which we still denote \mathbf{u}_n) has a weak limit \mathbf{u}_0 . Let $\mathbf{u}(t)$ be the solution of (1.1) with the initial data $\mathbf{u}(T_0) = \mathbf{u}_0$. Let $T < T_0$. In view of (3.57), for large n the sequence \mathbf{u}_n satisfies the assumptions of Corollary A.6 on the time interval $[T, T_0]$, hence $\mathbf{u}_n(T) \rightharpoonup \mathbf{u}(T)$. Passing to the weak limit in (3.57) finishes the proof. \square

Remark 3.14. Note that only the instability component $a_1^+(t)$ is treated via a topological argument, whereas $a_2^+(t)$ is estimated directly. This depends heavily on the (incidental) fact that the bootstrap bound $e^{-\frac{3}{2}\kappa|t|}$ is asymptotically smaller than $e^{-\nu|t|}$. Were it not the case, we would have to use a topological argument based on the Brouwer fixed point theorem, as in the work of Côte, Martel and Merle [10].

4. BUBBLE-ANTIBUBBLE FOR THE RADIAL CRITICAL YANG-MILLS EQUATION

4.1. Notation. In this section we denote $\|v\|_{L^2(r_1 \leq r \leq r_2)}^2 := 2\pi \int_{r_1}^{r_2} |v(r)|^2 r dr$. If r_1 or r_2 is not precised, then it should be understood that $r_1 = 0$, resp. $r_2 = +\infty$. The corresponding scalar product is denoted $\langle v, w \rangle := 2\pi \int_0^{+\infty} v(r) \cdot w(r) r dr$.

Recall that $\|v\|_{\mathcal{H}}^2 := 2\pi \int_0^{+\infty} (|\partial_r v(r)|^2 + |\frac{2}{r}v(r)|^2) r dr$. A change of variables shows that $v(r) \in \mathcal{H} \Leftrightarrow v(e^x) \in H^1(\mathbb{R})$, in particular $\|v\|_{L^\infty} \lesssim \|v\|_{\mathcal{H}}$ (this change of variables, very helpful in proving coercivity lemmas, can be found in [18]). Another useful way of understanding the space \mathcal{H} is to consider the transformation $\tilde{v}(e^{i\theta}r) := e^{2i\theta}v(r)$, which is an isometric embedding of \mathcal{H} in $\dot{H}^1(\mathbb{R}^2; \mathbb{R}^2)$, whose image is given by 2-equivariant functions in $\dot{H}^1(\mathbb{R}^2; \mathbb{R}^2)$. Let \mathcal{H}^* be the dual space of \mathcal{H} for

the pairing $\langle \cdot, \cdot \rangle$. The embedding just described identifies \mathcal{H}^* with the 2-equivariant distributions in $\dot{H}^{-1}(\mathbb{R}^2; \mathbb{R}^2)$.

We denote $X^0 := L^2 \cap \mathcal{H}$ and $X^1 := \{v \in \mathcal{H} : \partial_r v \in \mathcal{H} \text{ and } \frac{1}{r}v \in \mathcal{H}\}$. The generators of the \mathcal{H} -critical and the L^2 -critical scale change will be denoted respectively $\Lambda := r\partial_r$ and $\Lambda_0 := 1 + r\partial_r$.

4.2. Linearized equation and formal computation. Linearizing $-\partial_r^2 u - \frac{1}{r}\partial_r u + \frac{4}{r^2}u(1-u)(1-\frac{1}{2}u)$ around $u = W$ we obtain the operator

$$L := -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}(2 - 6(W(r) - 1)^2) = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}(4 - 6\Lambda W).$$

We study solutions behaving like $\mathbf{u}(t) \simeq -\mathbf{W} + \mathbf{W}_{\lambda(t)}$ with $\lambda(t) \rightarrow 0$ as $t \rightarrow -\infty$. As in Subsection 2.2, we expand

$$\mathbf{u}(t) = -\mathbf{W} + \mathbf{U}_{\lambda(t)}^{(0)} + b(t) \cdot \mathbf{U}_{\lambda(t)}^{(1)} + b(t)^2 \cdot \mathbf{U}_{\lambda(t)}^{(2)},$$

with $b(t) = \lambda'(t)$, $\mathbf{U}^{(0)} := (W, 0)$ and $\mathbf{U}^{(1)} := (0, -\Lambda W)$. This gives

$$(4.1) \quad \partial_t^2 u(t) = -b'(t)(\Lambda W)_{\lambda(t)} + \frac{b(t)^2}{\lambda(t)}(\Lambda_0 \Lambda W)_{\lambda(t)} + \text{lot}.$$

Let us restrict our attention to the region $r \leq \sqrt{\lambda(t)}$. We will see that the region $r \geq \sqrt{\lambda(t)}$ will not have much influence on the dynamics of our system. For $r \leq \sqrt{\lambda}$ we have $W \ll W_\lambda$, hence

$$4u(1-u)(1-\frac{1}{2}u) \simeq 4W_\lambda(1-W_\lambda)(1-\frac{1}{2}W_\lambda) + (-4 + 6\Lambda W_\lambda)W + \text{lot}.$$

Since $\partial_r^2 W + \frac{1}{r}\partial_r W \simeq \frac{4}{r^2}W + \text{lot}$ for $r \leq \sqrt{\lambda}$, we get

$$\partial_r^2 u + \frac{1}{r}\partial_r u - \frac{4}{r^2}u(1-u)(1-\frac{1}{2}u) = -\frac{b^2}{\lambda}(LU^{(2)})_\lambda - \frac{1}{r^2}6W_\lambda \cdot W + \text{lot}.$$

We can further simplify this using the fact that $W(r) \simeq 2r^2$:

$$\partial_r^2 u + \frac{1}{r}\partial_r u - \frac{4}{r^2}u(1-u)(1-\frac{1}{2}u) = -\frac{b^2}{\lambda}(LU^{(2)})_\lambda - 12\Lambda W_\lambda + \text{lot},$$

thus (4.1) yields

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda}{b^2}(b' - 12\lambda)\Lambda W.$$

As in Subsection 2.2, we find that the best choice of the formal parameter equations is

$$\lambda'(t) = b(t), \quad b'(t) = \kappa^2 \lambda(t), \quad \text{with } \kappa := 2\sqrt{3}.$$

Remark 4.1. The main term of the interaction is *exactly* cancelled by the term $-b'\Lambda W_\lambda$ for our choice of the parameters. We have seen that it is not the case for the power nonlinearity and in the next section we will see that it is not the case either for the critical equivariant wave map equation.

4.3. Bounds on the error of the ansatz. Fix $\mathcal{Z} \in C_0^\infty((0, +\infty))$ such that

$$(4.2) \quad \int_0^{+\infty} \mathcal{Z}(r) \cdot \Lambda W(r) \frac{dr}{r} > 0.$$

By a direct computation we find $L(\frac{r^4}{(1+r^2)^2}) = \frac{r^2(3-r^2)}{(1+r^2)^3} = -\Lambda_0 \Lambda W$. Adding a suitable multiple of $\Lambda W(r)$, we obtain a rational function $Q(r)$ such that

$$(4.3) \quad LQ = -\Lambda_0 \Lambda W, \quad \int \mathcal{Z}(r) \cdot Q(r) \frac{dr}{r} = 0, \quad Q(r) \sim r^2 \text{ as } r \rightarrow 0, \quad Q(r) \sim 1 \text{ as } r \rightarrow +\infty.$$

For λ, μ satisfying (2.6) and (2.7) (naturally with $\kappa = 2\sqrt{3}$) we define the approximate solution by the formula

$$\begin{aligned}\varphi(t) &:= -W_{\mu(t)} + W_{\lambda(t)} + S(t), \\ \dot{\varphi}(t) &:= -b(t)\Lambda W_{\underline{\lambda(t)}},\end{aligned}$$

where

$$\begin{aligned}b(t) &:= e^{-\kappa|T|} + \kappa^2 \int_T^t \frac{\lambda(\tau)}{\mu(\tau)^2} d\tau, & \text{for } t \in [T, T_0], \\ S(t) &:= \chi \cdot b(t)^2 Q_{\lambda(t)}, & \text{for } t \in [T, T_0].\end{aligned}$$

From (4.3) we obtain

$$\|\chi \cdot Q_{\underline{\lambda}}\|_{\mathcal{H}} \lesssim \left(\int_0^{2/\lambda} 1 \cdot r dr \right)^{\frac{1}{2}} + \frac{1}{\lambda} \cdot \left(\int_0^{2/\lambda} ((1+r)^{-1})^2 r dr \right)^{\frac{1}{2}} \lesssim \sqrt{|t|} \cdot e^{\kappa|t|},$$

which implies that

$$\|S(t)\|_{\mathcal{H}} \ll e^{-\frac{3}{2}\kappa|t|}.$$

Since \mathcal{Z} has compact support, (2.3) implies, for sufficiently small λ ,

$$\int \mathcal{Z}_{\underline{\lambda}} \cdot S(t) \frac{dr}{r} = 0.$$

We denote $f(u) := -4u(1-u)(1-\frac{1}{2}u)$ and

$$\begin{aligned}\psi(t) &= (\psi(t), \dot{\psi}(t)) := \partial_t \varphi(t) - DE(\varphi(t)) \\ &= (\partial_t \varphi(t) - \dot{\varphi}(t), \partial_t \dot{\varphi}(t) - (\partial_r^2 \varphi(t) + \frac{1}{r} \partial_r \varphi(t) + \frac{1}{r^2} f(\varphi(t))))).\end{aligned}$$

We have $f'(u) = 2 - 6(u-1)^2$.

Remark 4.2. By a direct computation, $f'(W) = -4 + 6\Lambda W$. Thus the potential term of the linearized operator contains a non-localized part -4 and a localized part $6\Lambda W$. These terms need to be treated in different ways. This is a known issue coming from the fact that $f(u)$ is not really the nonlinearity, as it “hides” the linear part near the stable equilibria: $f(u) \simeq -4u$ near $u = 0$ and $f(u) \simeq 4(2-u)$ near $u = 2$. Sometimes it is convenient to subtract the linear part from f , but here we work simultaneously near $u = 0$ and near $u = 2$, so probably it will be simpler to keep f as it is. A similar remark could be made in the case of the equivariant wave map equation.

Lemma 4.3. *Suppose that for $t \in [T, T_0]$ there holds $|\lambda'(t)| \lesssim e^{-\kappa|t|}$ and $|\mu'(t)| \lesssim e^{-\kappa|t|}$. Then*

$$(4.4) \quad \|\psi(t) - \mu'(t) \frac{1}{\mu(t)} \Lambda W_{\mu(t)} + (\lambda'(t) - b(t)) \frac{1}{\lambda(t)} \Lambda W_{\lambda(t)}\|_{\mathcal{H}} \lesssim e^{-\frac{3}{2}\kappa|t|},$$

$$(4.5) \quad \|\dot{\psi}(t) - \frac{b(t)}{\lambda(t)} (\lambda'(t) - b(t)) \Lambda_0 \Lambda W_{\underline{\lambda(t)}}\|_{L^2} \lesssim e^{-\frac{3}{2}\kappa|t|},$$

$$(4.6) \quad \|(-\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} f'(\varphi(t))) \psi(t)\|_{\mathcal{H}^*} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Proof. The proof of (4.4) is the same as the proof of (2.17).

In order to prove (4.5), we treat separately the regions $r \leq \sqrt{\lambda}$ and $r \geq \sqrt{\lambda}$. We will show that

$$(4.7) \quad \left\| \frac{1}{r^2} (f(\varphi) - f(W_{\lambda}) + 2 \frac{r^2}{\mu^2} f'(W_{\lambda}) - b^2 f'(W_{\lambda}) Q_{\lambda}) \right\|_{L^2(r \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Since f is a polynomial of degree 3, we have

$$f(\varphi) = f(W_\lambda) + f'(W_\lambda)(-W_\mu + S) + \frac{1}{2}f''(W_\lambda)(-W_\mu + S)^2 + \frac{1}{6}f'''(W_\lambda)(-W_\mu + S)^3.$$

We treat all the terms one by one. We have $|W_\mu(r) - 2\frac{r^2}{\mu^2}| \lesssim r^4$. Since $|f'(W_\lambda)| \lesssim 1$, this implies

$$\left\| \frac{1}{r^2} f'(W_\lambda)(W_\mu - 2r^2) \right\|_{L^2} \lesssim \left(\int_0^{\sqrt{\lambda}} r^4 \cdot r dr \right)^{\frac{1}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Since r is small, we have $S(r) = b^2 Q_\lambda(r)$, and the corresponding term is subtracted in (4.7).

Next, notice that $|W_\mu| + |S| \lesssim r^2$. Since $|f''(W_\lambda)| \lesssim 1$, this implies

$$\left\| \frac{1}{r^2} f''(W_\lambda)(-W_\mu + S)^2 \right\|_{L^2} \lesssim \left(\int_0^{\sqrt{\lambda}} \left(\frac{1}{r^2} \cdot r^4 \right)^2 r dr \right)^{\frac{1}{2}} \sim \lambda^{\frac{3}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

The last term is estimated in a similar way. This finishes the proof of (4.7).

By a direct computation $\left| \left(\partial_r^2 + \frac{1}{r} \partial_r \right) W_\mu - \frac{8}{\mu^2} \right| \lesssim r^2$, hence

$$(4.8) \quad \left\| \left(\partial_r^2 + \frac{1}{r} \partial_r \right) W_\mu - \frac{8}{\mu^2} \right\|_{L^2(r \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

From (4.7), (4.8) and the definition of $\varphi(t)$ we have

$$\left\| \left(\left(\partial_r^2 + \frac{1}{r} \partial_r \right) \varphi + \frac{1}{r^2} f(\varphi) \right) - \left(-\frac{2}{\mu^2} f'(W_\lambda) - \frac{b^2}{\lambda} (LQ)_\Delta - \frac{8}{\mu^2} \right) \right\|_{L^2(r \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Using (4.3) and the relation $4 + f'(W) = 6\Lambda W$ we can rewrite this as

$$\left\| \left(\left(\partial_r^2 + \frac{1}{r} \partial_r \right) \varphi + \frac{1}{r^2} f(\varphi) \right) - \left(-\frac{12}{\mu^2} \Lambda W_\lambda + \frac{b^2}{\lambda} \Lambda_0 \Lambda W_\Delta \right) \right\|_{L^2(r \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|},$$

which is equivalent to (4.5), restricted to the region $r \leq \sqrt{\lambda}$.

Consider the region $r \geq \sqrt{\lambda}$. Developping f at $2 - W_\mu$ we get

$$f(\varphi) = f(2 - W_\mu) + f'(2 - W_\mu)(W_\lambda - 2 + S) + \frac{1}{2}f''(2 - W_\mu)(W_\lambda - 2 + S)^2 + \frac{1}{6}f'''(2 - W_\mu)(W_\lambda - 2 + S)^3.$$

From this and the relations $f(2 - W_\mu) = -f(W_\mu)$, $f'(2 - W_\mu) = f'(W_\mu)$, we obtain a pointwise bound

$$(4.9) \quad |f(\varphi) + f(W_\mu) + f'(W_\mu)(2 - W_\lambda)| \lesssim |S| + |2 - W_\lambda|^2.$$

Since $|S(r)| \lesssim b^2$, we have

$$(4.10) \quad \left\| \frac{1}{r^2} S \right\|_{L^2(r \geq \sqrt{\lambda})} \lesssim b^2 \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-4} r dr \right)^{\frac{1}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

There holds $|2 - W_\lambda(r)|^2 \lesssim \frac{\lambda^4}{r^4}$, hence

$$(4.11) \quad \left\| \frac{1}{r^2} |2 - W_\lambda(r)|^2 \right\|_{L^2(r \geq \sqrt{\lambda})} \lesssim \lambda^4 \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-12} r dr \right)^{\frac{1}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Since $|W_\mu - 2| \lesssim r^2$, there holds $|f'(W_\mu) + 4| \lesssim r^2$. We also have $|2 - W_\lambda| \lesssim \frac{\lambda^2}{r^2}$ and $|(2 - W_\lambda) - \frac{2\lambda^2}{r^2}| \lesssim \frac{\lambda^4}{r^4}$, hence

$$(4.12) \quad \begin{aligned} & \left\| \frac{1}{r^2} f'(W_\mu)(2 - W_\lambda) + \frac{8\lambda^2}{r^4} \right\|_{L^2(r \geq \sqrt{\lambda})} \lesssim \left\| \frac{\lambda^2}{r^2} + \frac{\lambda^4}{r^6} \right\|_{L^2(r \geq \sqrt{\lambda})} \\ & \lesssim \lambda^2 \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-4} r dr \right)^{\frac{1}{2}} + \lambda^4 \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-12} r dr \right)^{\frac{1}{2}} \lesssim e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

Inserting (4.10), (4.11) and (4.12) into (4.9) we obtain

$$(4.13) \quad \|f(\varphi) + f(W_\mu) - \frac{8\lambda^2}{r^4}\|_{L^2(r \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

A direct computation shows that $(\partial_r^2 + \frac{1}{r}\partial_r)W_\lambda(r) = -\frac{8\lambda^2}{r^4} + O(\frac{\lambda^4}{r^6} + \frac{\lambda^8}{r^{10}})$, hence

$$(4.14) \quad \|(\partial_r^2 + \frac{1}{r}\partial_r)W_\lambda + \frac{8\lambda^2}{r^4}\|_{L^2(r \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

We have $\partial_r S = \chi' b^2 \cdot Q_\lambda + \chi b^2 \cdot (Q_\lambda)'$ and $\partial_r^2 S = \chi'' b^2 \cdot Q_\lambda + 2\chi' b^2 \cdot (Q_\lambda)' + \chi b^2 \cdot (Q_\lambda)''$. There holds $|Q| \lesssim 1$, $|Q'| \lesssim \frac{1}{r}$ and $|Q''| \lesssim \frac{1}{r^2}$, which implies $|Q_\lambda| \lesssim 1$, $|(Q_\lambda)'| \lesssim \frac{1}{r}$ and $|(Q_\lambda)''| \lesssim \frac{1}{r^2}$. This gives

$$(4.15) \quad \begin{aligned} \|\chi b^2 \cdot (Q_\lambda)''\|_{L^2(r \geq \sqrt{\lambda})} + \|\frac{1}{r}\chi b^2 \cdot (Q_\lambda)'\|_{L^2(r \geq \sqrt{\lambda})} &\lesssim b^2 \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-4} r dr \right)^{\frac{1}{2}} \lesssim \frac{b^2}{\sqrt{\lambda}} \lesssim e^{-\frac{3}{2}\kappa|t|}, \\ \|\chi' b^2 \cdot (Q_\lambda)'\|_{L^2(r \geq \sqrt{\lambda})} + \|\chi'' b^2 \cdot Q_\lambda\|_{L^2(r \geq \sqrt{\lambda})} + \|\frac{1}{r}\chi' b^2 \cdot Q_\lambda\|_{L^2(r \geq \sqrt{\lambda})} &\lesssim b^2 \ll e^{-\frac{3}{2}\kappa|t|}, \end{aligned}$$

hence $\|(\partial_r^2 + \frac{1}{r}\partial_r)S\|_{L^2(r \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}$. Together with (4.13) and (4.14) this proves that

$$\|(\partial_r^2 + \frac{1}{r}\partial_r)\varphi + \frac{1}{r^2}f(\varphi)\|_{L^2(r \geq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}.$$

Since $\|\Lambda W_\lambda\|_{L^2(r \geq \sqrt{\lambda})} + \|\Lambda_0 \Lambda W_\lambda\|_{L^2(r \geq \sqrt{\lambda})} \lesssim (\int_{1/\sqrt{\lambda}}^{+\infty} \frac{1}{r^4} r dr)^{\frac{1}{2}} \lesssim \sqrt{\lambda}$, the other terms appearing in (4.5) are $\lesssim e^{-\frac{3}{2}\kappa|t|}$. This finishes the proof of (4.5).

The proof of (4.6) is very similar to the proof of (2.19), hence we will just indicate the differences. Since $\|W_\lambda\|_{L^\infty} + \|W_\mu\|_{L^\infty} \lesssim 1$, we have $|f'(-W_\mu + W_\lambda) - f'(W_\lambda)| \lesssim W_\mu$ and $|f'(-W_\mu + W_\lambda) - f'(W_\mu)| = |f'(W_\mu + (2 - W_\lambda)) - f'(W_\mu)| \lesssim 2 - W_\lambda$. Next, we check that $\|\frac{1}{r^2}W_\mu \cdot \Lambda W_\lambda\|_{L^1} \lesssim e^{-\frac{3}{2}\kappa|t|}$ (recall that $\mathcal{H} \subset L^\infty$, hence $L^1(\mathbb{R}^2) \subset \mathcal{H}^*$). To do this, we consider separately $r \leq 1$ and $r \geq 1$:

$$\begin{aligned} \|\frac{1}{r^2}W_\mu \cdot \Lambda W_\lambda\|_{L^1(r \leq 1)} &\lesssim \|\Lambda W_\lambda\|_{L^1(r \leq 1)} = \lambda^2 \|\Lambda W\|_{L^1(r \leq 1/\lambda)} \lesssim \lambda^2 |\log \lambda| \ll e^{-\frac{3}{2}\kappa|t|}, \\ \|\frac{1}{r^2}W_\mu \cdot \Lambda W_\lambda\|_{L^1(r \geq 1)} &\lesssim \|\Lambda W_\lambda\|_{L^\infty(r \geq 1)} = |\Lambda W(\frac{1}{\lambda})| \sim \lambda^2 \ll e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

Finally, we check that $\|\frac{1}{r^2}(2 - W_\lambda) \cdot \Lambda W_\mu\|_{L^1} \lesssim e^{-\frac{1}{2}\kappa|t|}$, again dividing into $r \leq 1$ and $r \geq 1$:

$$\begin{aligned} \|2 - W_\lambda\|_{L^1(r \leq 1)} &= \left\| \frac{1}{1 + (r/\lambda)^2} \right\|_{L^1(r \leq 1)} = \lambda^2 \left\| \frac{1}{1 + r^2} \right\|_{L^1(r \leq 1/\lambda)} \sim \lambda^2 |\log \lambda| \ll e^{-\frac{3}{2}\kappa|t|}, \\ \|2 - W_\lambda\|_{L^1(r \geq 1)} &= \frac{2\lambda^2}{1 + \lambda^2} \ll e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

This allows to conclude, since $\|f'(\varphi) - f'(-W_\mu + W_\lambda)\|_{L^\infty} \lesssim \|S\|_{L^\infty} \lesssim e^{-\frac{3}{2}\kappa|t|}$. \square

4.4. Modulation. Having defined the approximate solution $\varphi(t)$, we will now analyse exact solutions close to $\varphi(t)$. The initial data are

$$\mathbf{u}(T) = (-W + W_{\frac{1}{\kappa}e^{-\kappa|T|}}, -e^{-\kappa|T|}\Lambda W_{\frac{1}{\kappa}e^{-\kappa|T|}})$$

(there is no linear instability in the case of the Yang-Mills equation).

Similarly as in Subsection 3.2, we choose the modulation parameters $\lambda(t)$ and $\mu(t)$ which verify

$$\int \mathcal{Z}_\mu \cdot (u(t) - (-W_{\mu(t)} + W_{\lambda(t)})) \frac{dr}{r} = 0, \quad \int \mathcal{Z}_\lambda \cdot (u(t) - (-W_{\mu(t)} + W_{\lambda(t)})) \frac{dr}{r} = 0.$$

We define $\mathbf{g}(t)$ by

$$\mathbf{u}(t) = \boldsymbol{\varphi}(t) + \mathbf{g}(t).$$

It satisfies, cf. (3.14),

$$\left\langle \frac{1}{\lambda(t)} \mathcal{Z}_{\lambda(t)}, g(t) \right\rangle = 0, \quad \left| \left\langle \frac{1}{\mu(t)} \mathcal{Z}_{\mu(t)}, g(t) \right\rangle \right| \leq c \cdot e^{-\frac{3}{2}\kappa|t|}.$$

The functions $\lambda(t)$ and $\mu(t)$ are C^1 and

$$|\lambda'(t) - b(t)| + |\mu'(t)| \lesssim \|\mathbf{g}(t)\|_{\mathcal{H}} + c \cdot e^{-\frac{3}{2}\kappa|t|}$$

with $c > 0$ arbitrarily small, cf. (3.15).

4.5. Coercivity. Recall that $f'(W) = -4 + 6\Lambda W$. In the next lemma, it is useful to separate these two terms, see Remark 4.2.

Lemma 4.4. *There exist constants $c, C > 0$ such that*

- *for all $g \in \mathcal{H}$ there holds*

$$(4.16) \quad \begin{aligned} & \int_0^{+\infty} ((g')^2 + \frac{4}{r^2} g^2) r dr - \int_0^{+\infty} \frac{6}{r^2} \Lambda W g^2 r dr \\ & \geq c \int_0^{+\infty} ((g')^2 + \frac{4}{r^2} g^2) r dr - C \left(\int_0^{+\infty} \mathcal{Z} \cdot g \frac{dr}{r} \right)^2, \end{aligned}$$

- *if $r_1 > 0$ is large enough, then for all $g \in \mathcal{H}$ there holds*

$$(4.17) \quad \begin{aligned} & (1 - 2c) \int_0^{r_1} ((g')^2 + \frac{4}{r^2} g^2) r dr + c \int_{r_1}^{+\infty} ((g')^2 + \frac{4}{r^2} g^2) r dr \\ & - \int_0^{+\infty} \frac{6}{r^2} \Lambda W g^2 r dr \geq -C \left(\int_0^{+\infty} \mathcal{Z} \cdot g \frac{dr}{r} \right)^2, \end{aligned}$$

- *if $r_2 > 0$ is small enough, then for all $g \in \mathcal{H}$ there holds*

$$(4.18) \quad \begin{aligned} & (1 - 2c) \int_{r_2}^{+\infty} ((g')^2 + \frac{4}{r^2} g^2) r dr + c \int_0^{r_2} ((g')^2 + \frac{4}{r^2} g^2) r dr \\ & - \int_0^{+\infty} \frac{6}{r^2} \Lambda W g^2 r dr \geq -C \left(\int_0^{+\infty} \mathcal{Z} \cdot g \frac{dr}{r} \right)^2, \end{aligned}$$

Proof. Let $\tilde{g}(x) := g(e^x)$ and $\tilde{\mathcal{Z}}(x) := \mathcal{Z}(e^x)$. One computes that $f'(W(e^x)) = -4 + 6 \operatorname{sech}^2(x)$, hence (4.16) is equivalent to

$$(4.19) \quad \int_{\mathbb{R}} (\tilde{g}')^2 + (4 - 6 \operatorname{sech}^2) \tilde{g}^2 dx \geq c \int_{\mathbb{R}} ((\tilde{g}')^2 + \tilde{g}^2) dx - C \left(\int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot \tilde{g} dx \right)^2.$$

This quadratic form corresponds to the classical operator $-\frac{d^2}{dx^2} + (4 - 6 \operatorname{sech}^2)$, for which 0 is a simple discrete eigenvalue, with the eigenspace spanned by sech^2 . Decompose $\tilde{g} = a \operatorname{sech}^2 + g_1$, with $\int \operatorname{sech}^2 \cdot g_1 dx = 0$. From the Sturm-Liouville theory we obtain

$$\int_{\mathbb{R}} (\tilde{g}')^2 + (4 - 6 \operatorname{sech}^2) \tilde{g}^2 dx = \int_{\mathbb{R}} (g_1')^2 + (4 - 6 \operatorname{sech}^2) g_1^2 dx \gtrsim \int_{\mathbb{R}} g_1^2 r dr.$$

Let $\text{sech}^2 = b\tilde{\mathcal{Z}} + (\text{sech}^2)^\perp$ with $\int \tilde{\mathcal{Z}} \cdot (\text{sech}^2)^\perp dx = 0$. Since $\int \tilde{\mathcal{Z}} \cdot \text{sech}^2 dx > 0$, see (4.2), we have $\int_{\mathbb{R}} ((\text{sech}^2)^\perp)^2 dx < \int_{\mathbb{R}} (\text{sech}^2)^2 dx$, hence

$$\begin{aligned} \int g_1^2 dx &= a^2 \int (\text{sech}^2)^2 dx - 2a \int \text{sech}^2 \cdot \tilde{g} dx + \int \tilde{g}^2 dx \\ &= a^2 \int (\text{sech}^2)^2 dx - 2ab \int \tilde{\mathcal{Z}} \cdot \tilde{g} dx - 2a \int (\text{sech}^2)^\perp \cdot \tilde{g} dx + \int \tilde{g}^2 dx \\ &\geq c \int \tilde{g}^2 dx - C \left(\int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot \tilde{g} dx \right)^2, \end{aligned}$$

which implies (4.19).

With the same change of variable, (4.17) and (4.18) will follow once we prove that

$$(4.20) \quad \begin{aligned} &(1 - 2c) \int_{|x| \leq R} ((g')^2 + 4g^2) dx + c \int_{|x| \geq R} ((g')^2 + 4g^2) dx \\ &- \int_{\mathbb{R}} 6 \text{sech}^2 g^2 dx \geq -C \left(\int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot g dx \right)^2, \end{aligned}$$

provided that R is large enough. To this end, take $\tilde{\chi}(x) := \chi(\frac{2x}{R})$ and $\tilde{h} := \tilde{\chi} \cdot \tilde{g}$. Since $\tilde{\mathcal{Z}}$ has compact support and R is large, we have $\int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot \tilde{h} dx = \int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot \tilde{g} dx$. By a standard integration by parts we get $\int_{\mathbb{R}} (\tilde{h}')^2 dx = \int_{\mathbb{R}} \tilde{\chi}^2 (\tilde{g}')^2 dx + \int_{\mathbb{R}} \frac{1}{2} ((\tilde{\chi}')^2 - \tilde{\chi} \tilde{\chi}'') \tilde{g}^2 dx$. We notice that $|(\tilde{\chi}')^2 - \tilde{\chi} \tilde{\chi}''| \lesssim R^{-2}$ is small, in particular for any $c > 0$ there holds $\int_{\mathbb{R}} \chi^2 (\tilde{g}')^2 dx \geq \int_{\mathbb{R}} (\tilde{h}')^2 dx - \frac{c}{2} \int ((\tilde{g}')^2 + 4\tilde{g}^2) x dx$, if R is large enough. Applying (4.19) with \tilde{h} instead of \tilde{g} and $3c$ instead of c we obtain

$$\begin{aligned} &(1 - 3c) \int_{|x| \leq R} ((\tilde{g}')^2 + 4\tilde{g}^2) dx - \int_{\mathbb{R}} 6 \text{sech}^2 \tilde{\chi}^2 \tilde{g}^2 dx \\ &\geq (1 - 3c) \int_{\mathbb{R}} \chi^2 ((\tilde{g}')^2 + 4\tilde{g}^2) dx - \int_{\mathbb{R}} 6 \text{sech}^2 \tilde{\chi}^2 \tilde{g}^2 dx \\ &\geq (1 - 3c) \int_{\mathbb{R}} ((\tilde{h}')^2 + 4\tilde{h}^2) dx - \frac{c}{2} \int_{\mathbb{R}} ((\tilde{g}')^2 + 4\tilde{g}^2) dx - \int_{\mathbb{R}} 6 \text{sech}^2 \tilde{h}^2 dx \\ &\geq -\frac{c}{2} \int_{\mathbb{R}} ((\tilde{g}')^2 + 4\tilde{g}^2) dx - C \left(\int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot \tilde{h} dx \right)^2 \geq -\frac{c}{2} \int_{\mathbb{R}} ((\tilde{g}')^2 + 4\tilde{g}^2) dx - C \left(\int_{\mathbb{R}} \tilde{\mathcal{Z}} \cdot \tilde{g} dx \right)^2. \end{aligned}$$

But $6 \text{sech}^2 \leq 6 \text{sech}^2 \tilde{\chi}^2 + 2c$ if R is large enough, and (4.20) follows. \square

Lemma 4.5. *There exists a constant $\eta > 0$ such that if $\frac{\lambda}{\mu} < \eta$ and $\|\mathbf{U} - (-\mathbf{W}_\mu + \mathbf{W}_\lambda)\|_{\mathcal{E}} < \eta$, then for all $\mathbf{g} \in \mathcal{E}$ there holds*

$$\langle D^2 E(\mathbf{U}) \mathbf{g}, \mathbf{g} \rangle + \left(\int \frac{1}{\lambda} \mathcal{Z}_\lambda \cdot \mathbf{g} \frac{dr}{r} \right)^2 + \left(\int \frac{1}{\mu} \mathcal{Z}_\mu \cdot \mathbf{g} \frac{dr}{r} \right)^2 \gtrsim \|\mathbf{g}\|_{\mathcal{E}}^2.$$

\square

The proof is a modification of the proof of Lemma 3.9 and will be skipped.

4.6. Definition of the mixed energy-virial functional.

Lemma 4.6. *For any $c > 0$ and $R > 0$ there exists a function $q(r) = q_{c,R}(r) \in C^{3,1}((0, +\infty))$ with the following properties:*

- (P1) $q(r) = \frac{1}{2}r^2$ for $r \leq R$,
- (P2) there exists $\tilde{R} > 0$ (depending on c and R) such that $q(r) \equiv \text{const}$ for $r \geq \tilde{R}$,
- (P3) $|q'(r)| \lesssim r$ and $|q''(r)| \lesssim 1$ for all $r > 0$, with constants independent of c and R ,
- (P4) $q''(r) \geq -c$ and $\frac{1}{r}q'(r) \geq -c$, for all $r > 0$,

(P5) $(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr})^2 q(r) \leq c \cdot r^{-2}$, for all $r > 0$,

(P6) $|r(\frac{q'(r)}{r})'| \leq c$, for all $r > 0$.

Proof. It suffices to prove the result for $R = 1$ since the function $q_R(r) := R^2 q(\frac{r}{R})$ satisfies the listed properties if and only if $q(r)$ does.

First we define $q_0(r)$ by the formula

$$q_0(r) := \begin{cases} \frac{1}{2} \cdot r^2 & r \leq 1 \\ \frac{1}{2}r^2 + c_1(\frac{1}{2}(r-1)^2 - \log(r)\frac{r^2-1}{4}) & r \geq 1, \end{cases}$$

with c_1 small. A direct computation shows that for $r > 1$ we have

$$(4.21) \quad \begin{aligned} q_0'(r) &= r(1 - \frac{c_1 \log r}{2}) + c_1(\frac{3}{4}r - 1 + \frac{1}{4r}), \\ q_0''(r) &= (1 - \frac{c_1 \log r}{2}) + c_1(\frac{1}{4} - \frac{1}{4r^2}), \\ q_0'''(r) &= -c_1 \frac{r^2 - 1}{2r^3}, \\ (\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr})^2 q_0(r) &= -\frac{c_1}{r^3}. \end{aligned}$$

In particular $q_0(1) = \frac{1}{2}$, $q_0'(1) = 1$, $q_0''(1) = 1$ and $q_0'''(1) = 0$, hence $q_0 \in C^{3,1}$.

Let $R_0 := e^{2/c_1}$. From (4.21) it follows that $q_0(r)$ verifies all the listed properties except for (P2) for $r \leq R_0$. Let $e_j(r) := \frac{1}{j!} r^j \cdot \chi(r)$ for $j \in \{1, 2, 3\}$, where χ is a standard cut-off function. We define

$$q(r) := \begin{cases} q_0(r) & r \leq R_0 \\ q_0(R_0) + \sum_{j=1}^3 q_0^{(j)}(R_0) \cdot R_0^j \cdot e_j(-1 + R_0^{-1}r) & r \geq R_0. \end{cases}$$

We will show that $q(r)$ has all the required properties if c_1 is small enough. Indeed, it is clear that $q(r) \in C^{3,1}((0, +\infty))$. It follows from (4.21) that $|q_0^{(j)}(R_0)| \lesssim c_1 R_0^{2-j}$ for $j \in \{1, 2, 3\}$. For $r > R_0$ and $k \in \{1, 2, 3, 4\}$ we have

$$q^{(k)}(r) = \sum_{j=1}^3 q_0^{(j)}(R_0) \cdot R_0^{j-k} e_j^{(k)}(-1 + R_0^{-1}r) \Rightarrow |q^{(k)}(r)| \lesssim c_1 R_0^{2-k}.$$

Since $q(r) \equiv \text{const}$ for $r \geq 3R_0$, we obtain (P1)–(P6). □

We define the operators $A(\lambda)$ and $A_0(\lambda)$ as follows:

$$\begin{aligned} [A(\lambda)h](r) &:= q'(\frac{r}{\lambda}) \cdot h'(r), \\ [A_0(\lambda)h](r) &:= (\frac{1}{2\lambda} q''(\frac{r}{\lambda}) + \frac{1}{2r} q'(\frac{r}{\lambda}))h(r) + q'(\frac{r}{\lambda}) \cdot h'(r). \end{aligned}$$

Lemma 4.7. *The operators $A(\lambda)$ and $A_0(\lambda)$ have the following properties:*

- the families $\{A(\lambda) : \lambda > 0\}$, $\{A_0(\lambda) : \lambda > 0\}$, $\{\lambda \partial_\lambda A(\lambda) : \lambda > 0\}$ and $\{\lambda \partial_\lambda A_0(\lambda) : \lambda > 0\}$ are bounded in $\mathcal{L}(\mathcal{H}; L^2)$, with the bound depending on the choice of the function $q(r)$,
- for all $\lambda > 0$ and $h_1, h_2 \in X^1$ there holds

$$(4.22) \quad \begin{aligned} &|\langle A(\lambda)h_1, \frac{1}{r^2}(f(h_1 + h_2) - f(h_1) - f'(h_1)h_2) \rangle + \langle A(\lambda)h_2, \frac{1}{r^2}(f(h_1 + h_2) - f(h_1) + 4h_2) \rangle| \\ &\leq \frac{c_0}{\lambda} ((\|h_1\|_{\mathcal{H}}^2 + 1)\|h_2\|_{\mathcal{H}}^2 + \|h_2\|_{\mathcal{H}}^4), \end{aligned}$$

with a constant c_0 arbitrarily small,

- for all $h \in X^1$ there holds

$$(4.23) \quad \langle A_0(\lambda)h, (\partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2})h \rangle \leq \frac{c_0}{\lambda} \|h\|_{\mathcal{H}}^2 - \frac{2\pi}{\lambda} \int_0^{R\lambda} ((\partial_r h)^2 + \frac{4}{r^2} h^2) r dr,$$

- assuming (2.7), for any $c_0 > 0$ there holds

$$(4.24) \quad \|\Lambda_0 \Lambda W_{\lambda(t)} - A_0(\lambda(t)) \Lambda W_{\lambda(t)}\|_{L^2} \leq c_0,$$

$$(4.25) \quad \|\dot{\varphi}(t) + b(t) \cdot A(\lambda(t))\varphi(t)\|_{L^\infty} \leq c_0,$$

$$(4.26) \quad \left| \int_0^{+\infty} \frac{1}{2} \left(q''\left(\frac{r}{\lambda}\right) + \frac{\lambda}{r} q'\left(\frac{r}{\lambda}\right) \right) \frac{1}{r^2} (f(\varphi + g) - f(\varphi) + 4g) g r dr \right. \\ \left. - \int_0^{+\infty} \frac{1}{r^2} (f'(W_\lambda) + 4) g^2 r dr \right| \leq c_0 C_0^2 e^{-3\kappa|t|},$$

provided that the constant R in the definition of $q(r)$ is chosen large enough.

Remark 4.8. The condition $\partial_r h_1, \frac{1}{r}h_1, \partial_r h_2, \frac{1}{r}h_2 \in \mathcal{H}$ is required only to ensure that the left hand side of (3.40) is well defined, but it does not appear on the right hand side of the estimate. Note also that in (4.22), (4.23) and (4.26) we extract the linear part of f , see Remark 4.2.

Proof. The proof of the first point is the same as in Lemma 3.12.

In (4.22), without loss of generality we may assume that $h_1, h_2 \in C_0^\infty((0, +\infty))$ and that $\lambda = 1$. From the definition of $A(\lambda)$ we have

$$\begin{aligned} & \langle A(\lambda)h_1, \frac{1}{r^2}(f(h_1 + h_2) - f(h_1) - f'(h_1)h_2) \rangle + \langle A(\lambda)h_2, \frac{1}{r^2}(f(h_1 + h_2) - f(h_1) + 4h_2) \rangle \\ &= \int_0^{+\infty} q'h'_1 \cdot \frac{1}{r^2}(f(h_1 + h_2) - f(h_1) - f'(h_1)) + q'h'_2 \cdot \frac{1}{r^2}(f(h_1 + h_2) - f(h_1) + 4h_2) r dr \\ &= \int_0^{+\infty} \frac{q'}{r} \left(\frac{d}{dr} F(h_1 + h_2) - \frac{d}{dr} F(h_1) - h_2 \cdot \frac{d}{dr} f(h_1) - f(h_1) \cdot \frac{d}{dr} h_2 + 2 \frac{d}{dr} (h_2^2) \right) dr \\ &= \int_0^{+\infty} r \left(\frac{q'}{r} \right)' \cdot \frac{1}{r^2} (F(h_1 + h_2) - F(h_1) - f(h_1) \cdot h_2 + 2h_2^2) r dr. \end{aligned}$$

Using (P6) in Lemma 4.6 and the elementary inequality $|F(h_1 + h_2) - F(h_1) - f(h_1)h_2| \lesssim |h_1|^2 |h_2|^2 + |h_1|^4$ we get (4.22).

Note that as a part of this computation, we obtain

$$(4.27) \quad |\langle A(\lambda)h, \frac{1}{r^2}h \rangle| \leq \frac{c_0}{\lambda} \int_0^{+\infty} \frac{1}{r^2} h^2 r dr.$$

For $r \leq R\lambda$ there holds $\frac{1}{2\lambda}q''(\frac{r}{\lambda}) + \frac{1}{2r}q'(\frac{r}{\lambda}) = \frac{1}{\lambda}$ and for all r , thanks to (P4), there holds $\frac{1}{2\lambda}q''(\frac{r}{\lambda}) + \frac{1}{2r}q'(\frac{r}{\lambda}) \geq -\frac{c_0}{\lambda}$, hence

$$(4.28) \quad \langle (\frac{1}{2\lambda}q''(\frac{r}{\lambda}) + \frac{1}{2r}q'(\frac{r}{\lambda}))h, \frac{1}{r^2}h \rangle \geq \frac{2\pi}{\lambda} \int_0^{R\lambda} \frac{1}{r^2} h^2 r dr - \frac{c_0}{\lambda} \int_0^{+\infty} \frac{1}{r^2} h^2 r dr.$$

Taking the sum of (4.27) and (4.28) we obtain

$$(4.29) \quad \langle A_0(\lambda)h, \frac{1}{r^2}h \rangle \geq -\frac{c_0}{\lambda} \int_0^{+\infty} \frac{1}{r^2} h^2 r dr + \frac{2\pi}{\lambda} \int_0^{R\lambda} \frac{1}{r^2} h^2 r dr$$

(c_0 has changed, but is still small).

Using identity (3.46) with $N = 2$ we obtain, cf. the proof of (3.41),

$$(4.30) \quad \langle A_0(\lambda)h, (\partial_r^2 + \frac{1}{r}\partial_r)h \rangle \leq \frac{c_0}{\lambda} \int_0^{+\infty} (\partial_r h)^2 r dr - \frac{2\pi}{\lambda} \int_{r \leq R\lambda} (\partial_r h)^2 r dr.$$

Taking the difference of (4.30) and (4.29) we obtain (3.41).

The proofs of (4.24) and (4.25) are similar to the proofs of (3.42) and (3.43) respectively. Instead of (3.48) we prove that $\|bA(\lambda)W_\mu\|_{L^\infty} \leq \frac{c_0}{3}$, which follows from (P2) and (P3). We skip the details.

The proof of (4.26) is close to the proof of (3.44). Note that it is crucial that $f'(W_\lambda) + 4$ vanishes at infinity. \square

For $t \in [T, T_0]$ we define:

- the *nonlinear energy functional*

$$\begin{aligned} I(t) &:= \int \frac{1}{2} |\dot{g}(t)|^2 + \frac{1}{2} |\nabla g(t)|^2 - \frac{1}{r^2} (F(\varphi(t) + g(t)) - F(\varphi(t)) - f(\varphi(t))g(t)) dx \\ &= E(\varphi(t) + \mathbf{g}(t)) - E(\varphi(t)) - \langle DE(\varphi(t), \mathbf{g}(t)) \rangle, \end{aligned}$$

- the *localized virial functional*

$$J(t) := \int \dot{g}(t) \cdot A_0(\lambda(t))g(t) dx,$$

- the *mixed energy-virial functional*

$$H(t) := I(t) + b(t)J(t).$$

4.7. Energy estimates via the mixed energy-virial functional. The remaining part of the proof is almost identical to Subsection 3.5. We will indicate the few differences.

Instead of (3.55), we obtain now

$$I'(t) \simeq -b(\lambda' - b) \cdot \langle A_0(\lambda)\Lambda W_{\underline{\lambda}}, \dot{g} \rangle - b \cdot \langle A(\lambda)g, \frac{1}{r^2} (f(\varphi + g) - f(\varphi) + 4g) \rangle.$$

As in the proof of Lemma 3.13, we have

$$\begin{aligned} (bJ)'(t) &\simeq b \int \dot{g} \cdot A_0(\lambda)(\dot{g} - \psi) r dr + b \int ((\partial_r^2 + \frac{1}{r}\partial_r)g \\ &\quad + \frac{1}{r^2}(f(\varphi + g) - f(\varphi)) - \dot{\psi}) \cdot A_0(\lambda)g r dr \\ &= b \int \dot{g} \cdot A_0(\lambda)(\dot{g} - \psi) r dr + b \int ((\partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2})g \\ &\quad + \frac{1}{r^2}(f(\varphi + g) - f(\varphi) + 4g) - \dot{\psi}) \cdot A_0(\lambda)g r dr, \end{aligned}$$

where we recognize the terms appearing in (4.23) and (4.26). The rest of the proof applies without change. Theorem 3 follows from the argument given in Subsection 3.6.

5. BUBBLE-ANTIBUBBLE FOR THE EQUIVARIANT CRITICAL WAVE MAP EQUATION

5.1. Notation. We use similar notation as in Section 4, with a slight modification in the definition of the norm \mathcal{H} :

$$\|v\|_{\mathcal{H}}^2 := 2\pi \int_0^{+\infty} (|\partial_r v(r)|^2 + |\frac{k}{r}v(r)|^2) r dr.$$

The transformation $\tilde{v}(e^{i\theta}r) := e^{ki\theta}v(r)$ is an isometric embedding of \mathcal{H} in $\dot{H}^1(\mathbb{R}^2; \mathbb{R}^2)$, whose image is given by k -equivariant functions in $\dot{H}^1(\mathbb{R}^2; \mathbb{R}^2)$.

5.2. Linearized equation and formal computation. Linearizing $-\partial_r^2 u - \frac{1}{r}\partial_r u + \frac{k^2}{2r^2}\sin(2u)$ around $u = W$ we obtain the operator

$$L := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}\cos(4\arctan(r^k)) = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}\left(1 - \frac{8}{(r^k + r^{-k})^2}\right).$$

It has a one-dimensional kernel spanned by ΛW . We fix $\mathcal{Z} \in C_0^\infty((0, +\infty))$ such that

$$\int_0^{+\infty} \mathcal{Z}(r) \cdot \Lambda W(r) \frac{dr}{r} > 0.$$

Lemma 5.1. *For all $V(r) \in C^\infty((0, +\infty))$ such that $\int_0^{+\infty} \Lambda W(r) \cdot V(r) r dr = 0$, $|V(r)| \lesssim r^k$ for small r and $|V(r)| \lesssim r^{-k}$ for large r , then there exists a function $U(r) \in C^\infty((0, +\infty))$ such that*

$$(5.1) \quad LU = V,$$

$$(5.2) \quad |U(r)| \lesssim r^k, \quad |\partial_r U(r)| \lesssim r^{k-1}, \quad |\partial_r^2 U(r)| \lesssim r^{k-2} \quad \text{for } r \text{ small},$$

$$(5.3) \quad |U(r)| \lesssim r^{-k}, \quad |\partial_r U(r)| \lesssim r^{-k-1}, \quad |\partial_r^2 U(r)| \lesssim r^{-k-2} \quad \text{for } r \text{ large},$$

$$(5.4) \quad \int \mathcal{Z}(r) \cdot U(r) \frac{dr}{r} = 0.$$

Proof. It is easy to check that the operator L factorizes as follows:

$$(5.5) \quad L = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}\left(1 - \frac{8}{(r^k + r^{-k})^2}\right) = \left(-\partial_r - \frac{1}{r} - \frac{\Lambda W'(r)}{\Lambda W(r)}\right)\left(\partial_r - \frac{\Lambda W'(r)}{\Lambda W(r)}\right),$$

hence we can invert it explicitly using twice the variation of constants formula. Define $U_1 \in C^\infty((0, +\infty))$ by

$$U_1(r) := \frac{1}{r\Lambda W(r)} \int_0^r V(\rho) \Lambda W(\rho) \rho d\rho.$$

It solves the equation $\left(-\partial_r - \frac{1}{r} - \frac{\Lambda W'(r)}{\Lambda W(r)}\right)U_1(r) = V(r)$. Since $|V(r)| \lesssim r^k$ and $\Lambda W(r) \sim r^k$ for small r , we have

$$(5.6) \quad |U_1(r)| \lesssim r^{-1-k+k+k+1+1} = r^{k+1}, \quad \text{small } r.$$

From the crucial assumption $\int_0^{+\infty} V(\rho) \Lambda W(\rho) \rho d\rho = 0$ we get

$$(5.7) \quad |U_1(r)| = \left| \frac{1}{r\Lambda W(r)} \int_r^{+\infty} V(\rho) \Lambda W(\rho) \rho d\rho \right| \lesssim r^{-k+1}, \quad \text{large } r.$$

From the differential equation we get also $|\partial_r U_1(r)| \lesssim r^k$ for small r and $|\partial_r U_1(r)| \lesssim r^{-k}$ for large r . Now we define $U \in C^\infty((0, +\infty))$ by the formula

$$U(r) := \Lambda W(r) \int_0^r \frac{U_1(\rho)}{\Lambda W(\rho)} d\rho.$$

It solves $\left(\partial_r - \frac{\Lambda W'(r)}{\Lambda W(r)}\right)U(r) = U_1(r)$, hence (5.5) yields (5.1). Using (5.6) and (5.7), one can check that $|U(r)| \lesssim r^{k+2}$ for small r and $|U(r)| \lesssim r^{-k+2}$ for large r . The differential equations yield $|\partial_r U(r)| \lesssim r^{k+1}$ and $|\partial_r^2 U(r)| \lesssim r^k$ for small r , as well as $|\partial_r U(r)| \lesssim r^{-k+1}$ and $|\partial_r^2 U(r)| \lesssim r^{-k}$ for large r . Adding to U a suitable multiple of ΛW we obtain (5.4). Since $|\Lambda W(r)| \lesssim r^k$, $|\partial_r \Lambda W(r)| \lesssim r^{k-1}$, $|\partial_r^2 \Lambda W(r)| \lesssim r^{k-2}$ for small r and $|\Lambda W(r)| \lesssim r^{-k}$, $|\partial_r \Lambda W(r)| \lesssim r^{-k-1}$, $|\partial_r^2 \Lambda W(r)| \lesssim r^{-k-2}$ for large r , (5.2) and (5.3) still hold. \square

We study solutions behaving like $\mathbf{u}(t) \simeq -\mathbf{W} + \mathbf{W}_{\lambda(t)}$ with $\lambda(t) \rightarrow 0$ as $t \rightarrow -\infty$. We expand

$$\mathbf{u}(t) = -\mathbf{W} + \mathbf{U}_{\lambda(t)}^{(0)} + b(t) \cdot \mathbf{U}_{\lambda(t)}^{(1)} + b(t)^2 \cdot \mathbf{U}_{\lambda(t)}^{(2)},$$

with $b(t) = \lambda'(t)$, $\mathbf{U}^{(0)} := (W, 0)$ and $\mathbf{U}^{(1)} := (0, -\Lambda W)$. As in Subsection 4.2, in the region $r \leq \sqrt{\lambda}$ we arrive at

$$\partial_r^2 u + \frac{1}{r} \partial_r u - \frac{k^2}{2r^2} \sin(2u) = -\frac{b^2}{\lambda} (LU^{(2)})_{\Delta} - \frac{1}{r^2} \cdot \frac{8k^2}{((r/\lambda)^k + (r/\lambda)^{-k})^2} W + \text{lot}.$$

Using the fact that $W(r) \sim 2r^k$ for small r we obtain

$$\partial_r^2 u + \frac{1}{r} \partial_r u - \frac{k^2}{2r^2} \sin(2u) = -\frac{b^2}{\lambda} (LU^{(2)})_{\Delta} - \frac{16k^2 r^{k-2}}{((r/\lambda)^k + (r/\lambda)^{-k})^2} + \text{lot},$$

thus, after rescaling,

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda}{b^2} (b' \Lambda W - \lambda^{k-1} \cdot \frac{16k^2 r^{k-2}}{(r^k + r^{-k})^2}).$$

It is not difficult to check (using for example the residue theorem) that

$$\int_0^{+\infty} \Lambda W(r) \cdot \frac{16k^2 r^{k-2}}{(r^k + r^{-k})^2} r dr = \frac{4k^2}{\pi} \sin\left(\frac{\pi}{k}\right) \cdot \int \Lambda W(r)^2 r dr,$$

hence the correct choice (that is, such that Lemma 5.1 allows to invert L) of the formal parameter equations is

$$\lambda'(t) = b(t), \quad b'(t) = \frac{4k^2}{\pi} \sin\left(\frac{\pi}{k}\right) \lambda(t)^{k-1} = \frac{k}{2} \left(\frac{2\kappa}{k-2}\right)^k \lambda(t)^{k-1},$$

where $\kappa := \frac{k-2}{2} \cdot \left(\frac{8k}{\pi} \sin\left(\frac{\pi}{k}\right)\right)^{\frac{1}{k}}$. This system has a solution

$$(\lambda_{\text{app}}(t), b_{\text{app}}(t)) = \left(\frac{k-2}{2\kappa} (\kappa|t|)^{-\frac{2}{k-2}}, (\kappa|t|)^{-\frac{k}{k-2}}\right), \quad t \leq T_0 < 0.$$

5.3. Bounds on the error of the ansatz. Let $I = [T, T_0]$ be the time interval, $T \leq T_0 < 0$, $|T_0|$ large. Let $\lambda(t)$ and $\mu(t)$ be C^1 functions on $[T, T_0]$ such that

$$(5.8) \quad \lambda(T) = \lambda_0, \quad \mu(T) = 1,$$

$$(5.9) \quad \frac{k-2}{4\kappa} (\kappa|t|)^{-\frac{2}{k-2}} \leq \lambda(t) \leq \frac{k-2}{\kappa} (\kappa|t|)^{-\frac{2}{k-2}}, \quad \frac{1}{2} \leq \mu(t) \leq 2,$$

with λ_0 satisfying $|\lambda_0 - \frac{k-2}{\kappa} (\kappa|T|)^{-\frac{2}{k-2}}| \lesssim |T|^{-\frac{5}{2(k-2)}}$. For technical reasons we cannot fix $\lambda_0 = \frac{k-2}{\kappa} (\kappa|T|)^{-\frac{2}{k-2}}$ as we did in the preceding sections. The value of λ_0 will be chosen later.

Let $P(r)$ and $Q(r)$ by the functions obtained in Lemma 5.1 for $V(r) = \frac{k}{2} \left(\frac{2\kappa}{k-2}\right)^k \Lambda W(r) - \frac{16k^2 r^{k-2}}{(r^k + r^{-k})^2}$ and $V(r) = -\Lambda_0 \Lambda W(r)$ respectively. We define the approximate solution by the formula

$$\begin{aligned} \varphi(t) &:= -W_{\mu(t)} + W_{\lambda(t)} + S(t), \\ \dot{\varphi}(t) &:= -b(t) \Lambda W_{\lambda(t)}. \end{aligned}$$

where

$$(5.10) \quad \begin{aligned} b(t) &:= \left(\frac{2\kappa}{k-2}\right)^{\frac{k}{2}} \lambda_0^{\frac{k}{2}} + \frac{k}{2} \left(\frac{2\kappa}{k-2}\right)^k \int_T^t \frac{\lambda(\tau)^{k-1}}{\mu(\tau)^k} d\tau, & \text{for } t \in [T, T_0], \\ S(t) &:= \frac{\lambda(t)^k}{\mu(t)^k} P_{\lambda(t)} + b(t)^2 Q_{\lambda(t)}, & \text{for } t \in [T, T_0]. \end{aligned}$$

The definition of $b(t)$ and (5.9) yield

$$b(t) \sim |t|^{-\frac{k}{k-2}},$$

with a constant depending only on k .

Since $P, Q \in \mathcal{H}$, there holds

$$(5.11) \quad \|S(t)\|_{\mathcal{H}} \lesssim |t|^{-\frac{2k}{k-2}}.$$

Note also that

$$\int \mathcal{Z}_{\underline{\lambda}} \cdot S(t) \frac{dr}{r} = 0.$$

We denote $f(u) := -\frac{k^2}{2} \sin(2u)$ (hence $f'(u) = -k^2 \cos(2u)$) and

$$\begin{aligned} \psi(t) &= (\psi(t), \dot{\psi}(t)) := \partial_t \varphi(t) - \text{DE}(\varphi(t)) \\ &= (\partial_t \varphi(t) - \dot{\varphi}(t), \partial_t \dot{\varphi}(t) - (\partial_r^2 \varphi(t) + \frac{1}{r} \partial_r \varphi(t) + \frac{1}{r^2} f(\varphi(t))))). \end{aligned}$$

We point out that usually, in the context of equivariant wave maps, $f(u)$ has the opposite sign. We chose the sign which is more coherent with the traditional notation for (1.1).

Lemma 5.2. *Suppose that for $t \in [T, T_0]$ there holds $|\lambda'(t)| \lesssim |t|^{-\frac{k}{k-2}}$ and $|\mu'(t)| \lesssim |t|^{-\frac{k}{k-2}}$. Then*

$$(5.12) \quad \begin{aligned} \|\psi(t) - \mu'(t) \frac{1}{\mu(t)} \Lambda W_{\mu(t)} + (\lambda'(t) - b(t)) \frac{1}{\lambda(t)} \Lambda W_{\lambda(t)}\|_{\mathcal{H}} &\lesssim |t|^{-\frac{2k-1}{k-2}}, \\ \|\dot{\psi}(t) - \frac{b(t)}{\lambda(t)} (\lambda'(t) - b(t)) \Lambda_0 \Lambda W_{\lambda(t)}\|_{L^2} &\lesssim |t|^{-\frac{2k-1}{k-2}}, \end{aligned}$$

$$(5.13) \quad \|(-\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} f'(\varphi(t))) \psi(t)\|_{\mathcal{H}^*} \lesssim |t|^{-\frac{2k-1}{k-2}}.$$

Proof. From the definition of ψ we get

$$\psi + \mu' \Lambda W_{\underline{\mu}} + (\lambda' - b) \Lambda W_{\underline{\lambda}} = -k \frac{\lambda^k}{\mu^{k+1}} \mu' P_{\lambda} + k \frac{\lambda^k}{\mu^k} \lambda' P_{\underline{\lambda}} - \frac{\lambda^k}{\mu^k} \lambda' \Lambda P_{\underline{\lambda}} + 2\lambda b b' Q_{\underline{\lambda}} - b^2 \lambda' \Lambda Q_{\underline{\lambda}}.$$

The first term has size $\lesssim |t|^{-\frac{3k}{k-2}} \ll |t|^{-\frac{2k-1}{k-2}}$ in \mathcal{H} . The other terms have size $\lesssim |t|^{-\frac{3k-2}{k-2}} \ll |t|^{-\frac{2k-1}{k-2}}$ in \mathcal{H} .

In order to prove (5.12), we treat separately the regions $r \leq \sqrt{\lambda}$ and $r \geq \sqrt{\lambda}$. First we will show that

$$(5.14) \quad \left\| \frac{1}{r^2} (f(\varphi) - f(W_{\lambda}) + 2 \frac{r^k}{\mu^k} f'(W_{\lambda}) - \frac{\lambda^k}{\mu^k} f'(W_{\lambda}) P_{\lambda} - b^2 f'(W_{\lambda}) Q_{\lambda}) \right\|_{L^2(r \leq \sqrt{\lambda})} \lesssim |t|^{-\frac{2k-1}{k-2}}.$$

We have an elementary pointwise inequality

$$(5.15) \quad |f(\varphi) - f(W_{\lambda}) - f'(W_{\lambda})(-W_{\mu} + S)| \lesssim |-W_{\mu} + S|^2,$$

with a constant depending only on k . We have $|W_{\mu}| \lesssim r^k$ and $|S| \lesssim (b^2 + \lambda^k) \cdot (\frac{r}{\lambda})^k \lesssim r^k$, hence

$$\left\| \frac{1}{r^2} |-W_{\mu} + S|^2 \right\|_{L^2(r \leq \sqrt{\lambda})} \lesssim \left(\int_0^{\sqrt{\lambda}} \left(\frac{1}{r^2} \cdot r^{2k} \right)^2 r dr \right)^{\frac{1}{2}} \sim \lambda^{\frac{2k-1}{2}} \lesssim |t|^{-\frac{2k-1}{k-2}},$$

which means that the right hand side in (5.15) is negligible. From the well-known fact that $|\arctan(z) - z| \lesssim |z|^3$ for small z we get $|W_{\mu}(r) - 2(r/\mu)^k| \lesssim r^{3k}$. This implies

$$\left\| \frac{1}{r^2} f'(W_{\lambda}) (W_{\mu} - 2 \frac{r^k}{\mu^k}) \right\|_{L^2} \lesssim \left(\int_0^{\sqrt{\lambda}} r^{6k-4} r dr \right)^{\frac{1}{2}} \lesssim \lambda^{\frac{3k-1}{2}} \lesssim |t|^{-\frac{3k-1}{k-2}} \ll |t|^{-\frac{2k-1}{k-2}}.$$

This proves (5.14) (see the definition of S).

We have $|W'_\mu(r) - \frac{2kr^{k-1}}{\mu^k}| \lesssim r^{3k-1}$ and $|W''_\mu(r) - \frac{2(k^2-k)r^{k-2}}{\mu^k}| \lesssim r^{3k-2}$ for small r , which implies $|(\partial_r^2 + \frac{1}{r}\partial_r)W_\mu(r) - \frac{2k^2r^{k-2}}{\mu^k}| \lesssim r^{3k-2}$, hence

$$(5.16) \quad \|(\partial_r^2 + \frac{1}{r}\partial_r)W_\mu - \frac{2k^2r^{k-2}}{\mu^k}\|_{L^2(r \leq \sqrt{\lambda})} \lesssim \left(\int_0^{\sqrt{\lambda}} r^{6k-4} r dr \right)^{\frac{1}{2}} \ll |t|^{-\frac{2k-1}{k-2}}.$$

Since $W''_\lambda(r) + \frac{1}{r}W'_\lambda(r) + \frac{1}{r^2}f(W_\lambda(r)) = 0$, from (5.14), (5.16) and the definition of $\varphi(t)$ we have

$$(5.17) \quad \begin{aligned} & \|((\partial_r^2 + \frac{1}{r}\partial_r)\varphi + \frac{1}{r^2}f(\varphi)) - (-\frac{2k^2r^{k-2}}{\mu^k} - \frac{2r^{k-2}}{\mu^k}f'(W_\lambda) - \frac{\lambda^{k-1}}{\mu^k}(LP)_\Delta \\ & \quad - \frac{b^2}{\lambda}(LQ)_\Delta)\|_{L^2(r \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|}. \end{aligned}$$

By the definition of P there holds $LP = -\frac{16k^2r^{k-2}}{(r^k+r^{-k})^2} + \frac{k}{2}(\frac{2\kappa}{k-2})^k \Lambda W = -2k^2r^{k-2} - 2r^{k-2}f'(W) + \frac{k}{2}(\frac{2\kappa}{k-2})^k \Lambda W$ and by the definition of Q there holds $LQ = -\Lambda_0 \Lambda W$, hence we can rewrite (5.17) as

$$\|((\partial_r^2 + \frac{1}{r}\partial_r)\varphi + \frac{1}{r^2}f(\varphi)) - (-\frac{k}{2}(\frac{2\kappa}{k-2})^k \frac{\lambda^{k-1}}{\mu^k} \Lambda W_\Delta + \frac{b^2}{\lambda} \Lambda_0 \Lambda W_\Delta)\|_{L^2(r \leq \sqrt{\lambda})} \lesssim e^{-\frac{3}{2}\kappa|t|},$$

which is precisely (5.12) restricted to $r \leq \sqrt{\lambda}$, cf. (2.25).

Consider the region $r \geq \sqrt{\lambda}$. We have $\varphi = (\pi - W_\mu) + (W_\lambda - \pi) + S$, hence elementary pointwise inequalities yield

$$|f(\varphi) - f(\pi - W_\mu) - f'(\pi - W_\mu)(W_\lambda - \pi + S)| \lesssim |W_\lambda - \pi + S|^2.$$

From this and the relations $f(\pi - W_\mu) = -f(W_\mu)$, $f'(\pi - W_\mu) = f'(W_\mu)$, we obtain a pointwise bound

$$(5.18) \quad |f(\varphi) + f(W_\mu) + f'(W_\mu)(\pi - W_\lambda)| \lesssim |S| + |\pi - W_\lambda|^2.$$

Since $|S(r)| \lesssim b^2 + \frac{\lambda^k}{\mu^k} \sim |t|^{-\frac{2k}{k-2}}$, we have

$$(5.19) \quad \|\frac{1}{r^2}S\|_{L^2(r \geq \sqrt{\lambda})} \lesssim |t|^{-\frac{2k}{k-2}} \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-4} r dr \right)^{\frac{1}{2}} \lesssim \frac{|t|^{-\frac{2k}{k-2}}}{\sqrt{\lambda}} \lesssim |t|^{-\frac{2k-1}{k-2}}.$$

There holds $|\pi - W_\lambda(r)|^2 = |\pi - 2 \arctan((\lambda/r)^k)|^2 = |2 \arctan((\lambda/r)^k)|^2 \lesssim \frac{\lambda^{2k}}{r^{2k}}$, hence

$$(5.20) \quad \|\frac{1}{r^2}|\pi - W_\lambda(r)|^2\|_{L^2(r \geq \sqrt{\lambda})} \lesssim \lambda^{2k} \left(\int_{\sqrt{\lambda}}^{+\infty} r^{-4k-4} r dr \right)^{\frac{1}{2}} \sim \lambda^{2k} \lambda^{-\frac{2k+1}{2}} \sim |t|^{-\frac{2k-1}{k-2}}.$$

Recall that $f'(W_\mu) = -k^2(1 - 8((r/\mu)^k + (r/\mu)^{-k})^{-2})$, hence $|f'(W_\mu) + k^2| \lesssim r^k$. We also have (by a standard asymptotic expansion of \arctan) $|\pi - W_\lambda| \lesssim \frac{\lambda^k}{r^k}$ and $|\pi - W_\lambda - 2\frac{\lambda^k}{r^k}| \lesssim \frac{\lambda^{3k}}{r^{3k}}$, hence

$$(5.21) \quad \left\| \frac{1}{r^2}f'(W_\mu)(\pi - W_\lambda) + \frac{2k^2\lambda^k}{r^{k+2}} \right\|_{L^2(r \geq \sqrt{\lambda})} \lesssim \left\| \frac{\lambda^k}{r^2} + \frac{\lambda^{3k}}{r^{3k+2}} \right\|_{L^2(r \geq \sqrt{\lambda})} \lesssim |t|^{-\frac{2k-1}{k-2}}.$$

Inserting (5.19), (5.20) and (5.21) into (5.18) we obtain

$$(5.22) \quad \left\| \frac{1}{r^2}f(\varphi) + \frac{1}{r^2}f(W_\mu) - \frac{2k^2\lambda^k}{r^{k+2}} \right\|_{L^2(r \geq \sqrt{\lambda})} \lesssim |t|^{-\frac{2k-1}{k-2}}.$$

A direct computation shows that for $r \geq \sqrt{\lambda}$ there holds $(\partial_r^2 + \frac{1}{r}\partial_r)W_\lambda(r) = -\frac{2k^2\lambda^k}{r^{k+2}} + O(\frac{\lambda^{3k}}{r^{3k+2}})$, hence

$$(5.23) \quad \|(\partial_r^2 + \frac{1}{r}\partial_r)W_\lambda + \frac{2k^2\lambda^k}{r^{k+2}}\|_{L^2(r \geq \sqrt{\lambda})} \lesssim \lambda^{\frac{3k-1}{2}} \ll |t|^{-\frac{2k-1}{k-2}}.$$

The same computation as in (4.15) yields $\|(\partial_r^2 + \frac{1}{r}\partial_r)S\|_{L^2(r \geq \sqrt{\lambda})} \lesssim \frac{b^2}{\sqrt{\lambda}} + \frac{\lambda^k}{\sqrt{\lambda}} \lesssim |t|^{-\frac{2k-1}{k-2}}$. Together with (5.22) and (5.23) this proves that

$$\|(\partial_r^2 + \frac{1}{r}\partial_r)\varphi + \frac{1}{r^2}f(\varphi)\|_{L^2(r \geq \sqrt{\lambda})} \lesssim |t|^{-\frac{2k-1}{k-2}}.$$

Since $\|\Lambda W_\Delta\|_{L^2(r \geq \sqrt{\lambda})} + \|\Lambda_0 \Lambda W_\Delta\|_{L^2(r \geq \sqrt{\lambda})} \lesssim (\int_{1/\sqrt{\lambda}}^{+\infty} \frac{1}{r^{2k}} r dr)^{\frac{1}{2}} \lesssim \lambda^{\frac{k-1}{2}}$, the other terms appearing in (5.12) are $\lesssim |t|^{-\frac{3k-3}{k-2}} \ll |t|^{-\frac{2k-1}{k-2}}$. This finishes the proof of (5.12).

The proof of (5.13) is very similar to the proof of (2.19), so we will not give all the details. We have $|\frac{\lambda'-b}{\lambda}| \lesssim |t|^{-1}$ and $|\frac{\mu'}{\mu}| \lesssim |t|^{-1}$, hence it suffices to check that $\|\frac{1}{r^2}(f'(\varphi) - f'(W_\lambda))\Lambda W_\lambda\|_{\mathcal{H}^*} \lesssim |t|^{-\frac{k+1}{k-2}}$ and $\|\frac{1}{r^2}(f'(\varphi) - f'(W_\mu))\Lambda W_\mu\|_{\mathcal{H}^*} \lesssim |t|^{-\frac{k+1}{k-2}}$, which boils down to $\|\frac{1}{r^2}W_\mu \cdot \Lambda W_\lambda\|_{L^1} \lesssim |t|^{-\frac{k+1}{k-2}}$ and $\|\frac{1}{r^2}(\pi - W_\lambda) \cdot \Lambda W_\mu\|_{L^1} \lesssim |t|^{-\frac{k+1}{k-2}}$, see the proof of (4.6). In both cases we treat separately $r \leq 1$ and $r \geq 1$:

$$\begin{aligned} \|\frac{1}{r^2}W_\mu \cdot \Lambda W_\lambda\|_{L^1(r \leq 1)} \|r^{k-2}\Lambda W_\lambda\|_{L^1(r \leq 1)} &= \lambda^k \|\Lambda W\|_{L^1(r \leq 1/\lambda)} \lesssim \lambda^k |\log \lambda| \ll |t|^{-\frac{k+1}{k-2}}, \\ \|\frac{1}{r^2}W_\mu \cdot \Lambda W_\lambda\|_{L^1(r \geq 1)} \|\Lambda W\|_{L^\infty(r \geq 1)} &\lesssim \lambda^k \ll |t|^{-\frac{k+1}{k-2}}, \\ \|\frac{1}{r^2}(\pi - W_\lambda) \cdot \Lambda W_\mu\|_{L^1(r \leq 1)} &\lesssim \|r^{k-2} \arctan((\lambda/r)^k)\|_{L^1(r \leq 1)} \\ &= \lambda^k \|r^{k-2} \arctan(r^{-k})\|_{L^1(r \leq 1/\lambda)} \lesssim \lambda^k |\log \lambda| \ll |t|^{-\frac{k+1}{k-2}}, \\ \|\frac{1}{r^2}(\pi - W_\lambda) \cdot \Lambda W_\mu\|_{L^1(r \geq 1)} &\lesssim \|\pi - W_\lambda\|_{L^\infty(r \geq 1)} \lesssim \lambda^k \ll |t|^{-\frac{k+1}{k-2}}. \end{aligned}$$

□

5.4. Modulation. As in Subsection 3.2, we define $\mathbf{g}(t) := \mathbf{u}(t) - \varphi(t)$ with modulation parameters $\lambda(t)$ and $\mu(t)$ which satisfy

$$\begin{aligned} \langle \frac{1}{\lambda(t)} \mathcal{Z}_{\lambda(t)}, \mathbf{g}(t) \rangle &= 0, \quad |\langle \frac{1}{\mu(t)} \mathcal{Z}_{\mu(t)}, \mathbf{g}(t) \rangle| \lesssim c |t|^{-\frac{k+1}{k-2}}, \\ (5.24) \quad |\lambda'(t) - b(t)| + |\mu'(t)| &\lesssim \|\mathbf{g}(t)\|_{\mathcal{E}} + c |t|^{-\frac{k+1}{k-2}}, \end{aligned}$$

with a constant c arbitrarily small. The initial data are

$$(5.25) \quad \mathbf{u}(T) = (-W + W_{\lambda_0}, -(\frac{2\kappa}{k-2})^{\frac{k}{2}} \lambda_0^{\frac{k}{2}} \Lambda W_{\lambda_0}),$$

where the value of λ_0 is close to $\frac{k-2}{2\kappa}(\kappa|T|)^{-\frac{2}{k-2}}$. The initial value of $\lambda(t)$ in the modulation equations is thus $\lambda(T) = \lambda_0$, in accordance with (5.8), and $b(t)$ is defined by (5.10). Note that $b(T) = (\frac{2\kappa}{k-2})^{\frac{k}{2}} \lambda_0^{\frac{k}{2}}$ is close to $(\kappa|T|)^{-\frac{k}{k-2}}$.

The equivalent of Proposition 3.3 can be formulated as follows.

Proposition 5.3. *There exist constants $C_0 > 0$ and $T_0 < 0$ with the following property Let $T < T_1 < T_0$ and suppose that $\mathbf{u}(t) = \varphi(t) + \mathbf{g}(t) \in C([T, T_1]; X^1 \times X^0)$ is a solution of (1.4) with initial data (5.25) such that for $t \in [T, T_1]$ condition (5.9) is satisfied and*

$$(5.26) \quad \|\mathbf{g}(t)\|_{\mathcal{E}} \leq C_0 |t|^{-\frac{k+1}{k-2}}.$$

Then for $t \in [T, T_1]$ there holds

$$(5.27) \quad \|\mathbf{g}(t)\|_{\mathcal{E}} \leq \frac{1}{2}C_0|t|^{-\frac{k+1}{k-2}},$$

$$(5.28) \quad |\mu(t) - 1| \lesssim C_0|t|^{-\frac{3}{k-2}},$$

$$(5.29) \quad \left| \lambda'(t) - \left(\frac{2\kappa}{k-2} \right)^{\frac{k}{2}} \lambda(t)^{\frac{k}{2}} \right| \lesssim C_0|t|^{-\frac{k+1}{k-2}}.$$

Note that, unlike in Proposition 3.3, we do not obtain an improvement of estimates on $\lambda(t)$. Indeed, we will need a shooting argument in order to control $\lambda(t)$.

Proof of Theorem 2 assuming Proposition 5.3. We consider the functions

$$\beta_j(t) := \frac{k-2}{2\kappa}(\kappa|t|)^{-\frac{2}{k-2}} + j|t|^{-\frac{5}{2(k-2)}},$$

for $j \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$. They will serve as barriers for $\lambda(t)$ in the topological argument.

Let T_n be a decreasing sequence converging to $-\infty$. For n large and $\lambda_0 \in \mathcal{A} := [\beta_{-1}(T), \beta_1(T)]$, let $\mathbf{u}_n^{\lambda_0}(t) : [T_n, T_+] \rightarrow \mathcal{E}$ denote the solution of (1.4) with initial data (5.25). We will prove that there exists λ_0 such that $T_+ > T_0$ and $\mathbf{u} = \mathbf{u}_n^{\lambda_0}$ verifies for $t \in [T_n, T_0]$ the estimates (5.27), (5.28) and

$$(5.30) \quad \beta_{-1}(t) \leq \lambda(t) \leq \beta_1(t).$$

Suppose that this is not the case. For each $\lambda_0 \in \mathcal{A}$, let $T_1 = T_1(\lambda_0)$ be the last time such that these three estimates hold for $t \in [T_n, T_1]$. Proposition 5.3 implies that only (5.30) can possibly break down. Let \mathcal{A}_+ be the set of $\lambda_0 \in \mathcal{A}$ such that $\lambda(T_1) = \beta_1(T_1)$ and \mathcal{A}_- the set of λ_0 such that $\lambda(T_1) = \beta_{-1}(T_1)$. We will prove that \mathcal{A}_+ and \mathcal{A}_- are open sets. They are also non-empty since $\beta_{-1}(T) \in \mathcal{A}_-$ and $\beta_1(T) \in \mathcal{A}_+$. The end of the proof is then the same as in Subsection 3.6.

The key step is to show that if for a certain $T_2 \in [T_n, T_1]$ we have $\lambda(T_2) > \beta_{\frac{1}{2}}(T_2)$, then $\lambda_0 \in \mathcal{A}_+$. Suppose this is not the case. Then there exists the smallest $T_3 > T_2$ such that $\lambda(T_3) = \beta_{\frac{1}{2}}(T_3)$. This implies that

$$(5.31) \quad \lambda'(T_3) \leq \beta'_{\frac{1}{2}}(T_3).$$

On the other hand, (5.29) yields

$$\begin{aligned} \lambda'(T_3) &\geq \left(\frac{2\kappa}{k-2} \right)^{\frac{k}{2}} (\lambda(T_3))^{\frac{k}{2}} - C_0|T_3|^{-\frac{k+1}{k-2}} = \left(\frac{2\kappa}{k-2} \right)^{\frac{k}{2}} (\beta_{\frac{1}{2}}(T_3))^{\frac{k}{2}} - C_0|T_3|^{-\frac{k+1}{k-2}} \\ &= |T_3|^{-\frac{k}{k-2}} \left(\kappa^{-\frac{2}{k-2}} + \frac{\kappa}{k-2} |T_3|^{-\frac{1}{2(k-2)}} \right)^{\frac{k}{2}} - C_0|T_3|^{-\frac{k+1}{k-2}} \\ &\geq |T_3|^{-\frac{k}{k-2}} \left(\kappa^{-\frac{k}{k-2}} + \frac{k}{2} \cdot \kappa^{-1} \frac{\kappa}{k-2} |T_3|^{-\frac{1}{2(k-2)}} \right) - (C + C_0)|T_3|^{-\frac{k+1}{k-2}} \\ &= (\kappa|T_3|)^{-\frac{k}{k-2}} + \frac{k}{2(k-2)} |T_3|^{-\frac{2k+1}{2(k-2)}} - (C + C_0)|T_3|^{-\frac{k+1}{k-2}}. \end{aligned}$$

Since $\beta'_{\frac{1}{2}}(T_3) = (\kappa|T_3|)^{-\frac{k}{k-2}} + \frac{5}{4(k-2)} |T_3|^{-\frac{2k+1}{2(k-2)}}$ and $\frac{k}{2(k-2)} > \frac{5}{4(k-2)}$ for $k \geq 3$, (5.31) is impossible if $|T_3| > |T_0|$ is large enough.

Analogously, if for a certain $T_2 \in [T_n, T_1]$ we have $\lambda(T_2) < \beta_{-\frac{1}{2}}(T_2)$, then $\lambda_0 \in \mathcal{A}_-$. Hence, if $\lambda_0 \in \mathcal{A}_+$, then $\lambda(t) \geq \beta_{-\frac{1}{2}}(t)$ for $t \in [T_n, T_1]$ and $\lambda(T_1) = \beta_1(T_1)$. By continuity, if $|\tilde{\lambda}_0 - \lambda_0|$ is sufficiently small, then the solution corresponding to $\tilde{\lambda}_0$ will satisfy $\tilde{\lambda}(t) > \beta_{\frac{1}{2}}(t)$ at some point, hence $\tilde{\lambda}_0 \in \mathcal{A}_+$. This proves that \mathcal{A}_+ is open. Analogously, \mathcal{A}_- is open. \square

5.5. Coercivity. Recall that $f'(W) = -k^2 \left(1 - \frac{8}{(r^k + r^{-k})^2}\right)$.

Lemma 5.4. *There exist constants $c, C > 0$ such that*

- *for all $g \in \mathcal{H}$ there holds*

$$\begin{aligned} & \int_0^{+\infty} \left((g')^2 + \frac{k^2}{r^2} g^2 \right) r dr - \int_0^{+\infty} \frac{k^2}{r^2} \cdot \frac{8}{(r^k + r^{-k})^2} g^2 r dr \\ & \geq c \int_0^{+\infty} \left((g')^2 + \frac{k^2}{r^2} g^2 \right) r dr - C \left(\int_0^{+\infty} \mathcal{Z} \cdot g \frac{dr}{r} \right)^2, \end{aligned}$$

- *if $r_1 > 0$ is large enough, then for all $g \in \mathcal{H}$ there holds*

$$\begin{aligned} (5.32) \quad & (1 - 2c) \int_0^{r_1} \left((g')^2 + \frac{k^2}{r^2} g^2 \right) r dr + c \int_{r_1}^{+\infty} \left((g')^2 + \frac{k^2}{r^2} g^2 \right) r dr \\ & - \int_0^{+\infty} \frac{k^2}{r^2} \cdot \frac{8}{(r^k + r^{-k})^2} g^2 r dr \geq -C \left(\int_0^{+\infty} \mathcal{Z} \cdot g \frac{dr}{r} \right)^2, \end{aligned}$$

- *if $r_2 > 0$ is small enough, then for all $g \in \mathcal{H}$ there holds*

$$\begin{aligned} & (1 - 2c) \int_{r_2}^{+\infty} \left((g')^2 + \frac{k^2}{r^2} g^2 \right) r dr + c \int_0^{r_2} \left((g')^2 + \frac{k^2}{r^2} g^2 \right) r dr \\ & - \int_0^{+\infty} \frac{k^2}{r^2} \cdot \frac{8}{(r^k + r^{-k})^2} g^2 r dr \geq -C \left(\int_0^{+\infty} \mathcal{Z} \cdot g \frac{dr}{r} \right)^2, \end{aligned}$$

Proof. A change of variable $\tilde{g}(x) := g(e^{x/k})$, $\tilde{\mathcal{Z}}(x) := \mathcal{Z}(e^{x/k})$ reduces the problem to the study of the quadratic form associated with the classical operator $-\frac{d^2}{dx^2} + (1 - 2 \operatorname{sech}^2)$ and it suffices to repeat the proof of Lemma 4.4. \square

Lemma 4.5 applies verbatim to the case under consideration.

5.6. Bootstrap. We use the operators $A(\lambda)$ and $A_0(\lambda)$ from Subsection 4.6. We define $I(t)$, $J(t)$ and $H(t)$ by the same formulas as in Subsection 4.6.

Lemma 5.5. *The operators $A(\lambda)$ and $A_0(\lambda)$ have the following properties:*

- *for all $\lambda > 0$ and $h_1, h_2 \in X^1$ there holds*

$$\begin{aligned} (5.33) \quad & \left| \left\langle A(\lambda) h_1, \frac{1}{r^2} (f(h_1 + h_2) - f(h_1) - f'(h_1) h_2) \right\rangle \right. \\ & \left. + \left\langle A(\lambda) h_2, \frac{1}{r^2} (f(h_1 + h_2) - f(h_1) + k^2 h_2) \right\rangle \right| \leq \frac{c_0}{\lambda} \cdot \|h_2\|_{\mathcal{H}}^2, \end{aligned}$$

with a constant c_0 arbitrarily small,

- *for all $h \in X^1$ there holds*

$$(5.34) \quad \left\langle A_0(\lambda) h, \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right) h \right\rangle \leq \frac{c_0}{\lambda} \|h\|_{\mathcal{H}}^2 - \frac{2\pi}{\lambda} \int_0^{R\lambda} \left((\partial_r h)^2 + \frac{k^2}{r^2} h^2 \right) r dr,$$

- assuming (2.7), for any $c_0 > 0$ there holds

$$(5.35) \quad \|\Lambda_0 \Lambda W_{\underline{\lambda}(t)} - A_0(\lambda(t)) \Lambda W_{\lambda(t)}\|_{L^2} \leq c_0,$$

$$(5.36) \quad \|\dot{\varphi}(t) + b(t) \cdot A(\lambda(t)) \varphi(t)\|_{L^\infty} \leq c_0 |t|^{-1},$$

$$(5.37) \quad \left| \int_0^{+\infty} \frac{1}{2} \left(q''\left(\frac{r}{\lambda}\right) + \frac{\lambda}{r} q'\left(\frac{r}{\lambda}\right) \right) \frac{1}{r^2} (f(\varphi + g) - f(\varphi) + k^2 g) g \, r \, dr \right. \\ \left. - \int_0^{+\infty} \frac{1}{r^2} (f'(W_\lambda) + k^2) g^2 \, r \, dr \right| \leq c_0 C_0^2 |t|^{-\frac{2k+2}{k-2}},$$

provided that the constant R in the definition of $q(r)$ is chosen large enough. \square

The proof is almost identical to the proof of Lemma 4.7 and we will skip it.

Lemma 5.6. *Let $c_1 > 0$. If C_0 is sufficiently large, then there exists a function $q(x)$ and $T_0 < 0$ with the following property. If $T_1 < T_0$ and (5.9), (5.26) hold for $t \in [T, T_1]$, then for $t \in [T, T_1]$ there holds*

$$(5.38) \quad H'(t) \leq c_1 \cdot C_0^2 |t|^{-\frac{3k}{k-2}}.$$

Proof. We follow the lines of the proof of Lemma 3.13. We have

$$I'(t) = \left\langle \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} f'(\varphi) \right) \psi, g \right\rangle - \langle \dot{\psi}, g \rangle - \left\langle \dot{\varphi}, \frac{1}{r^2} (f(\varphi + g) - f(\varphi) - f'(\varphi)g) \right\rangle.$$

The first term is $\lesssim C_0 |t|^{-\frac{3k}{k-2}}$, hence negligible (by enlarging C_0 if necessary). Inequality (5.12) implies that the second term can be replaced by $-\frac{b}{\lambda} (\lambda' - b) \langle \Lambda_0 \Lambda W_{\underline{\lambda}}, \dot{g} \rangle$, which in turn can be replaced by $-b(\lambda' - b) \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle$, thanks to (5.35). From (5.36) we infer that the third term can be replaced by $b \cdot \langle A(\lambda) \varphi, \frac{1}{r^2} (f(\varphi + g) - f(\varphi) - f'(\varphi)g) \rangle$ (indeed, $\left\| \frac{1}{r^2} (f(\varphi + g) - f(\varphi) - f'(\varphi)g) \right\|_{L^1} \lesssim \int_0^{+\infty} \frac{1}{r^2} g^2 \, r \, dr \lesssim \|g\|_{\mathcal{H}}^2$). Using formula (5.33) with $h_1 = \varphi$ and $h_2 = g$ we obtain

$$I'(t) \simeq -b(\lambda' - b) \cdot \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle - b \cdot \left\langle A(\lambda) g, \frac{1}{r^2} (f(\varphi + g) - f(\varphi) + k^2 g) \right\rangle.$$

As in the proof of Lemma 3.13, we obtain

$$(bJ)'(t) \simeq b(\lambda' - b) \cdot \langle A_0(\lambda) \Lambda W_{\underline{\lambda}}, \dot{g} \rangle \\ + b \int \left(\left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right) g + \frac{1}{r^2} (f(\varphi + g) - f(\varphi) + k^2 g) \right) \cdot A_0(\lambda) g \, r \, dr,$$

hence

$$H'(t) \simeq b \int \left(\left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right) g \right) \cdot A_0(\lambda) g \, r \, dr \\ - \frac{b}{\lambda} \int_0^{+\infty} \frac{1}{2} \left(q''\left(\frac{r}{\lambda}\right) + \frac{\lambda}{r} q'\left(\frac{r}{\lambda}\right) \right) \frac{1}{r^2} (f(\varphi + g) - f(\varphi) + k^2 g) g \, r \, dr.$$

and the conclusion follows from (5.34), (5.37) and (5.32). \square

Proof of Proposition 5.3. We first show (5.28). From (5.24) and (5.26) we obtain

$$(5.39) \quad |\mu(t) - 1| = |\mu(t) - \mu(T)| \lesssim \int_{-\infty}^t C_0 |t|^{-\frac{k+1}{k-2}} \, dt \lesssim C_0 |t|^{-\frac{3}{k-2}}.$$

Again from (5.24) and (5.26) we have $|\lambda'(t) - b(t)| \lesssim C_0 |t|^{-\frac{k+1}{k-2}}$. Multiplying by $b'(t) = \frac{k}{2} \cdot \left(\frac{2\kappa}{k-2} \right)^k \cdot \frac{\lambda(t)^{k-1}}{\mu(t)^k} \sim |t|^{-\frac{2k-2}{k-2}}$, cf. (5.10) and (5.9), we obtain $\left| \frac{d}{dt} (b(t)^2 - \left(\frac{2\kappa}{k-2} \right)^k \frac{\lambda(t)^k}{\mu(t)^k} \right| \lesssim C_0 |t|^{-\frac{3k-1}{k-2}}$.

Since $b(T) = \left(\frac{2\kappa}{k-2}\right)^{\frac{k}{2}} \cdot \lambda(T)^{\frac{k}{2}}$ and $\mu(T) = 1$, this yields $|b(t)^2 - \left(\frac{2\kappa}{k-2}\right)^k \frac{\lambda(t)^k}{\mu(t)^k}| \lesssim C_0 |t|^{-\frac{2k+1}{k-2}}$. But $b(t) + \left(\frac{2\kappa}{k-2}\right)^{\frac{k}{2}} \frac{\lambda(t)^{\frac{k}{2}}}{\mu(t)^{\frac{k}{2}}} \sim |t|^{\frac{k}{k-2}}$, see (2.7) and (2.9), hence

$$(5.40) \quad \left| b(t) - \left(\frac{2\kappa}{k-2}\right)^{\frac{k}{2}} \frac{\lambda(t)^{\frac{k}{2}}}{\mu(t)^{\frac{k}{2}}} \right| \lesssim C_0 |t|^{-\frac{k+1}{k-2}}.$$

Bound (5.39) implies that $\left| \frac{\lambda(t)^{\frac{k}{2}}}{\mu(t)^{\frac{k}{2}}} - \lambda(t)^{\frac{k}{2}} \right| \ll |t|^{-\frac{k+1}{k-2}}$, thus (5.40) yields $|\lambda'(t) - \left(\frac{2\kappa}{k-2}\right)^{\frac{k}{2}} \lambda(t)^{\frac{k}{2}}| \lesssim C_0 |t|^{-\frac{k+1}{k-2}}$, which is (5.29).

We turn to the proof of (5.27). From (5.11) the initial data at $t = T$ satisfy $\|\mathbf{g}(T)\|_{\mathcal{E}} \lesssim |T|^{-\frac{2k}{k-2}} \ll |T|^{-\frac{k+1}{k-2}}$, thus $H(T) \lesssim |T|^{-\frac{2k+2}{k-2}}$. If C_0 is large enough, then integrating (5.38) we get $H(t) \leq c \cdot C_0^2 |t|^{-\frac{2k+2}{k-2}}$, with a small constant c . Increasing C_0 if necessary and using the coercivity of H , we obtain (5.27). \square

APPENDIX A. CAUCHY THEORY

A.1. Persistence of regularity.

Proposition A.1. *Let $\mathbf{u} : (T_-, T_+) \rightarrow \mathcal{E}$ be the solution of (1.1) with the initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$. If $\mathbf{u}_0 \in X^1 \times H^1$, then $\mathbf{u} \in C((T_-, T_+); X^1 \times H^1) \cap C^1((T_-, T_+); \mathcal{E})$.* \square

The proof is classical, see [5, Chapter 5] for more general results in the case of the nonlinear Schrödinger equation. Analogous results hold in the case of equations (1.5) and (1.4).

We recall also the following fact from the Cauchy theory of (1.2) in the energy space. For $I \subset \mathbb{R}$ we denote $S(I) = L^{\frac{2(N+1)}{N-2}}(I; L^{\frac{2(N+1)}{N-2}})$ the Strichartz norm on the time interval I . For $t \in \mathbb{R}$ we denote $S(t)$ the linear wave propagator (this traditional collision of notation should not lead to misunderstanding).

Proposition A.2. *There exists $\eta > 0$ such that if $\|\mathbf{u}_0\|_{\mathcal{E}} \leq \eta$, then the solution of (1.1) with the initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$ is global and*

$$\sup_{t \in \mathbb{R}} \|\mathbf{u}(t)\|_{\mathcal{E}} \lesssim \inf_{t \in \mathbb{R}} \|\mathbf{u}(t)\|_{\mathcal{E}}.$$

Moreover, $\|u(t)\|_{S(\mathbb{R})} < +\infty$ and there exists $\mathbf{v}_+ \in \mathcal{E}$ such that

$$\lim_{t \rightarrow +\infty} \|\mathbf{u}(t) - S(t)\mathbf{v}_+\|_{\mathcal{E}} = 0$$

(analogously for $t \rightarrow -\infty$).

A.2. Profile decomposition and consequences. For details about the nonlinear profile decomposition for the critical wave equation we refer to [2] (the defocusing case), [13] (dimension $N = 3$) and [35] (any dimension). For the reader's convenience we recall the following result [35, Proposition 2.3].

Proposition A.3. *Let $\mathbf{u}_{0,n}$ be a bounded sequence in \mathcal{E} admitting a profile decomposition with profiles \mathbf{U}_L^j and parameters $\lambda_{j,n}, t_{j,n}$. Let \mathbf{U}^j be the corresponding nonlinear profiles and let θ_n be a sequence of positive numbers. Assume that*

$$\forall j \geq 1, n \geq 1, \quad \frac{\theta_n - t_{j,n}}{\lambda_{j,n}} < T_+(\mathbf{U}^j) \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \|\mathbf{U}^j\|_{S\left(\frac{-t_{j,n}}{\lambda_{j,n}}, \frac{\theta_n - t_{j,n}}{\lambda_{j,n}}\right)} < +\infty.$$

Let \mathbf{u}_n be the solution of (1.1) with the initial data $\mathbf{u}_n(0) = \mathbf{u}_{0,n}$. Then for n sufficiently large \mathbf{u}_n is defined on $[0, \theta_n]$ and

$$\mathbf{u}_n(t) = \sum_{j=1}^J \mathbf{U}^j\left(\frac{t-t_{j,n}}{\lambda_{j,n}}\right)_{\lambda_{j,n}} + \mathbf{w}_n^J(t) + \mathbf{r}_n^J(t), \quad \text{for all } J \in \mathbb{N} \text{ and } t \in [0, \theta_n],$$

with $\lim_{J \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sup_{t \in [0, \theta_n]} \|\mathbf{r}_n^J\|_{\mathcal{E}} = 0$. An analogous statement holds for $\theta \leq 0$.

For a corresponding result for the Yang-Mills equation and the equivariant wave maps, see [23].

Corollary A.4. *There exists a constant $\eta > 0$ such that the following holds. Let $\mathbf{u} : [t_0, T_+) \rightarrow \mathcal{E}$ be a maximal solution of (1.1) with $T_+ < +\infty$. Then for any compact set $K \subset \mathcal{E}$ there exists $\tau < T_+$ such that $\text{dist}(\mathbf{u}(t), K) > \eta$ for $t \in [\tau, T_+)$.*

Proof. Suppose for the sake of contradiction that there exists a sequence $T_n \rightarrow T_+$ such that for $\mathbf{u}_{0,n} := \mathbf{u}(T_n)$ there holds $\text{dist}(\mathbf{u}_{0,n}, K) \leq \eta$, hence $\mathbf{u}_{0,n} = \mathbf{k}_n + \mathbf{b}_n$ with $\mathbf{k}_n \in K$ and $\|\mathbf{b}_n\|_{\mathcal{E}} \leq \eta$. By taking a subsequence we can assume that $\mathbf{k}_n \rightarrow \mathbf{k}_0 \in K$ and that the sequence \mathbf{b}_n admits a profile decomposition, in particular $\mathbf{b}_n \rightarrow \mathbf{b}_0 \in \mathcal{E}$. This implies that $\mathbf{u}_{0,n}$ admits a profile decomposition with the first profile $\mathbf{U}^1 = \mathbf{k}_0 + \mathbf{b}_0$ (with parameters $\lambda_{1,n} = 1$ and $t_{1,n} = 0$) and all the other profiles small in the energy norm. Proposition A.3 leads to a contradiction. \square

Lemma A.5. *Let \mathbf{U} be a solution of (1.2) such that $\|\mathbf{U}(t)\|_{\mathcal{E}}$ is small (for any, hence for all t). Let t_n, λ_n be a sequence of parameters such that*

$$(A.1) \quad |\log(\lambda_n)| + \frac{|t_n|}{\lambda_n} \rightarrow +\infty.$$

Then for all $t \in \mathbb{R}$ there holds

$$(A.2) \quad \mathbf{U}\left(\frac{t-t_n}{\lambda_n}\right)_{\lambda_n} \rightarrow 0 \quad \text{in } \mathcal{E}.$$

Proof. It is sufficient to prove (A.2) for a subsequence of any subsequence, thus we can assume that $\frac{t-t_n}{\lambda_n} \rightarrow t_0 \in [-\infty, +\infty]$.

Suppose first that $t_0 \in (-\infty, +\infty)$, hence $\mathbf{U}\left(\frac{t-t_n}{\lambda_n}\right) \rightarrow \mathbf{U}(t_0)$. Extracting again a subsequence we can assume that $\lambda_n \rightarrow \lambda_0 \in [0, +\infty]$. Then (A.1) implies that $\lambda_0 = 0$ or $\lambda_0 = +\infty$, and in both cases we get (A.2).

Suppose that $t_0 = +\infty$ (the case $t_0 = -\infty$ is the same). Let $\mathbf{V}_+ \in \mathcal{E}$ be such that $\lim_{t \rightarrow +\infty} \|\mathbf{U}(t) - S(t)\mathbf{V}_+\|_{\mathcal{E}} = 0$. Again, we can assume that $\lambda_n \rightarrow \lambda_0 \in [0, +\infty]$. It is well known that $S(t)\mathbf{V}_+ \rightarrow 0$ as $t \rightarrow +\infty$, which settles the case $\lambda_0 \in (0, +\infty)$. In the case $\lambda_0 = 0$ or $\lambda_0 = +\infty$, notice that $S\left(\frac{t-t_n}{\lambda_n}\right)$ is a Fourier multiplier, hence the Fourier support of $(S\left(\frac{t-t_n}{\lambda_n}\right)\mathbf{V}_+)_{\lambda_n}$ is concentrated in an annulus of characteristic size λ_n^{-1} , which implies the weak convergence to 0 on the Fourier side. \square

Corollary A.6. *There exists a constant $\eta > 0$ such that the following holds. Let $K \subset \mathcal{E}$ be a compact set and let $\mathbf{u}_n : [T_1, T_2] \rightarrow \mathcal{E}$ be a sequence of solutions of (1.1) such that*

$$\text{dist}(\mathbf{u}_n(t), K) \leq \eta, \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [T_1, T_2].$$

Suppose that $\mathbf{u}_n(T_1) \rightarrow \mathbf{u}_0 \in \mathcal{E}$. Then the solution $\mathbf{u}(t)$ of (1.1) with the initial condition $\mathbf{u}(T_1) = \mathbf{u}_0$ is defined for $t \in [T_1, T_2]$ and

$$(A.3) \quad \mathbf{u}_n(t) \rightarrow \mathbf{u}(t), \quad \text{for all } t \in [T_1, T_2].$$

Proof. It suffices to prove (A.3) for a subsequence of any subsequence. Hence, we may assume that $\mathbf{u}_{0,n} := \mathbf{u}_n(T_1)$ admits a profile decomposition. As in the proof of Corollary A.4, we show that all the profiles except for $\mathbf{U}^1 = \mathbf{u}_0$ are small in the energy norm. Moreover, for $j > 1$ the sequence

of parameters $(\lambda_{j,n}, t_{j,n})$ is pseudo-orthogonal to $(\lambda_{1,n}, t_{1,n}) = (1, 0)$, which means precisely that $(\lambda_n, t_n) = (\lambda_{j,n}, t_{j,n})$ verifies (A.1). Proposition A.3 implies (A.3). \square

Remark A.7. Corollaries A.4 and A.6 hold for the Yang-Mills and wave map equations, with the same proofs.

Remark A.8. An important point of both results is that η is independent of K . Corollary A.4 states that a blow-up cannot happen at a small distance from a compact set. Corollary A.6 establishes sequential weak continuity of the flow in a neighbourhood of any compact set. Without this additional condition weak continuity is expected to fail, a counterexample being provided by type II blow-up solutions.

Remark A.9. Corollaries A.4 and A.6 are crucial ingredients of the arguments in Subsection 3.6. Using the nonlinear profile decomposition of Bahouri and Gérard [2] is nowadays a well-established method of attacking this type of questions in critical spaces. Note that [2] gave the first proof of the sequential weak continuity of the flow for the defocusing energy-critical wave equation.

REFERENCES

- [1] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.*, 55(9):269–296, 1976.
- [2] H. Bahouri and P. Gérard. High frequency approximation of solutions to critical nonlinear wave equations. *Amer. J. Math.*, 121(1):131–175, 1999.
- [3] P. Bizoń, T. Chmaj, and Z. Tabor. Formation of singularities for equivariant $2 + 1$ dimensional wave maps into the 2-sphere. *Nonlinearity*, 14(5):1041–1053, 2001.
- [4] A. Bulut, M. Czubak, D. Li, N. Pavlović, and X. Zhang. Stability and unconditional uniqueness of solutions for energy critical wave equations in high dimensions. *Comm. Part. Diff. Eq.*, 38(4):575–607, 2013.
- [5] T. Cazenave. *Semilinear Schrödinger Equations*, volume 10 of *Courant Lecture Notes in Mathematics*. AMS, 2003.
- [6] T. Cazenave, J. Shatah, and A. Tahvildar-Zadeh. Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 68(3):315–349, 1998.
- [7] R. Côte. On the soliton resolution for equivariant wave maps to the sphere. *Comm. Pure Appl. Math.*, 68(11):1946–2004, 2015.
- [8] R. Côte, C. E. Kenig, A. Lawrie, and W. Schlag. Characterization of large energy solutions of the equivariant wave map problem: I. *Amer. J. Math.*, 137(1):139–207, 2015.
- [9] R. Côte, C. E. Kenig, and F. Merle. Scattering below critical energy for the radial 4D Yang-Mills equation and for the 2D corotational wave map system. *Comm. Math. Phys.*, 284(1):203–225, 2008.
- [10] R. Côte, Y. Martel, and F. Merle. Construction of multi-soliton solutions for the L^2 -supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.*, 27(1):273–302, 2011.
- [11] R. Donninger, M. Huang, J. Krieger, and W. Schlag. Exotic blowup solutions for the u^5 focusing wave equation in \mathbb{R}^3 . *Michigan Math. J.*, 63(3):451–501, 2014.
- [12] R. Donninger and J. Krieger. Nonscattering solutions and blowup at infinity for the critical wave equation. *Math. Ann.*, 357(1):89–163, 2013.
- [13] T. Duyckaerts, C. E. Kenig, and F. Merle. Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation. *J. Eur. Math. Soc.*, 13(3):533–599, 2011.
- [14] T. Duyckaerts, C. E. Kenig, and F. Merle. Profiles of bounded radial solutions of the focusing, energy-critical wave equation. *Geom. Funct. Anal.*, 22(3):639–698, 2012.
- [15] T. Duyckaerts, C. E. Kenig, and F. Merle. Classification of the radial solutions of the focusing, energy-critical wave equation. *Camb. J. Math.*, 1(1):75–144, 2013.
- [16] T. Duyckaerts and F. Merle. Dynamics of threshold solutions for energy-critical wave equation. *Int. Math. Res. Pap. IMRP*, 2008.
- [17] J. Ginibre, A. Soffer, and G. Velo. The global Cauchy problem for the critical nonlinear wave equation. *J. Funct. Anal.*, 110:96–130, 1992.
- [18] S. Gustafson, K. Kang, and T.-P. Tsai. Schrödinger flow near harmonic maps. *Comm. Pure Appl. Math.*, 60(4):463–499, 2007.
- [19] M. Hillairet and P. Raphaël. Smooth type II blow up solutions to the four dimensional energy critical wave equation. *Anal. PDE*, 5(4):777–829, 2012.

- [20] R. van der Hout. On the nonexistence of finite time bubble trees in symmetric harmonic map heat flows from the disk to the 2-sphere. *J. Differential Equations*, 192(1):188–201, 2003.
- [21] J. Jendrej. Construction of type II blow-up solutions for the energy-critical wave equation in dimension 5. *Preprint*, arXiv:1503.05024, 2015.
- [22] J. Jendrej. Nonexistence of radial two-bubbles with opposite signs for the energy-critical wave equation. *Preprint*, arXiv:1510.03965, 2015.
- [23] H. Jia and C. E. Kenig. Asymptotic decomposition for semilinear wave and equivariant wave map equations. *Preprint*, arXiv:1503.06715, 2015.
- [24] C. E. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing nonlinear wave equation. *Acta Math.*, 201(2):147–212, 2008.
- [25] J. Krieger and W. Schlag. Full range of blow up exponents for the quintic wave equation in three dimensions. *J. Math Pures Appl.*, 101(6):873–900, 2014.
- [26] J. Krieger, W. Schlag, and D. Tataru. Renormalization and blow up for charge one equivariant critical wave maps. *Invent. Math.*, 171(3):543–615, 2008.
- [27] J. Krieger, W. Schlag, and D. Tataru. Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation. *Duke Math. J.*, 147(1):1–53, 2009.
- [28] A. Lawrie and S.-J. Oh. A refined threshold theorem for $(1+2)$ -dimensional wave maps into surfaces. *Comm. Math. Phys.*, 342(3):989–999, 2016.
- [29] Y. Martel. Asymptotic N -soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.*, 127(5):1103–1140, 2005.
- [30] Y. Martel and F. Merle. Construction of multi-solitons for the energy-critical wave equation in dimension 5. *Preprint*, arXiv:1504.01595, 2015.
- [31] Y. Martel and P. Raphaël. Strongly interacting blow up bubbles for the mass critical NLS. *Preprint*, arXiv:1512.00900, 2015.
- [32] F. Merle. Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Commun. Math. Phys.*, 129(2):223–240, 1990.
- [33] P. Raphaël and J. Szeftel. Existence and uniqueness of minimal mass blow up solutions to an inhomogeneous L^2 -critical NLS. *J. Amer. Math. Soc.*, 24(2):471–546, 2011.
- [34] I. Rodnianski and J. Sterbenz. On the formation of singularities in the critical $O(3)$ σ -model. *Ann. of Math.*, 172(1):187–242, 2010.
- [35] C. Rodriguez. Profiles for the radial focusing energy-critical wave equation in odd dimensions. *Preprint*, arXiv:1412.1388, 2014.
- [36] J. Shatah and M. Struwe. Well-posedness in the energy space for semilinear wave equations with critical growth. *Internat. Math. Res. Notices*, 7:303–309, 1994.
- [37] J. Shatah and M. Struwe. *Geometric Wave Equations*, volume 2 of *Courant Lecture Notes in Mathematics*. AMS, 2000.
- [38] J. Shatah and A. Tahvildar-Zadeh. On the Cauchy problem for equivariant wave maps. *Comm. Pure Appl. Math.*, 47(5):719–754, 1994.
- [39] M. Struwe. Equivariant wave maps in two space dimensions. *Comm. Pure Appl. Math.*, 56(7):815–823, 2003.
- [40] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl.*, 110(4):353–372, 1976.
- [41] P. Topping. An example of a nontrivial bubble tree in the harmonic map heat flow. In *Harmonic Morphisms, Harmonic Maps and Related Topics*. Chapman and Hall/CRC, 1999.

CMLS, ÉCOLE POLYTECHNIQUE, CNRS, UNIVERSITÉ PARIS-SACLAY, 91128 PALAISEAU CEDEX, FRANCE
 E-mail address: jacek.jendrej@polytechnique.edu